November 2025
M.Sc.
Third Semester
CORE – 10
MATHEMATICS
Course Code: MMAC 3.21
(Functional Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Let $J = [a, b]$ be a closed interval. Let X denotes the set of all real-valued functions $x, y, \ldots$ that are defined and continuous on J. Define a metric on X by $d(x, y) = \max_{t \in J} |x(t) - y(t)|$, the metric space is denoted by $C[a, b]$. Prove that the metric space $(C[a, b], d)$ is complete. 5

(b) Let X be the real vector space of all ordered pairs $x = (x_1, x_2)$ of real numbers. Show that each of the following maps $\|\cdot\| : X \to [0, \infty)$ defines a norm on X: 5

(i) The Euclidean (ℓ^2) norm: $\|x\|_2 := \sqrt{x_1^2 + x_2^2}$

(ii) The ℓ^∞ norm: $\|x\|_\infty := \max\{|x_1|, |x_2|\}$

(iii) For any $p \geq 1$, the ℓ^p norm: $\|x\|_p := \left(|x_1|^p + |x_2|^p\right)^{1/p}$

(c) Show that every compact subset of a metric space is closed and bounded. Give a counterexample to the converse in an infinite-dimensional normed space. 4

2. (a) Let ℓ^∞ be the space of bounded sequences with norm $\|x\|_\infty = \sup_n |x_n|$, and let $Y := \{x \in \ell^\infty : x_n = 0 \text{ for all but finitely many } n\}$. Prove that Y is a linear subspace of ℓ^∞ but is not closed in ℓ^∞. 4

(b) Let X be a normed vector space over $\mathbb{K}(\mathbb{R} \text{ or } \mathbb{C})$, and let $\{x_1, \ldots, x_n\} \subset X$ be linearly independent. Prove that there exists a constant $c > 0$ such that for all scalars $a_1, \ldots, a_n$, $\left\|\sum_{k=1}^{n} a_k x_k\right\| \geq c \sum_{k=1}^{n} |ak|$. Specialise to $X = \mathbb{R}^m$ with Euclidean norm $\|\cdot\|_2$ and the standard basis. Determine the largest constant c for which the inequality above holds: 6+4=10

(i) $m = 2,\ x_1 = (1,\ 0),\ x_2 = (0,\ 1)$

(ii) $m = 3,\ x_1 = (1,\ 0,\ 0),\ x_2 = (0,\ 1,\ 0),\ x_3 = (0,\ 0,\ 1)$

UNIT–II

3. (a) Let $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator between normed spaces. Prove:

(i) T is continuous $\Leftrightarrow T$ is bounded 5

(ii) If T is continuous at some $x_0 \in \mathcal{D}(T)$, then T is continuous on $\mathcal{D}(T)$ 4

(b) Let ℓ^∞ be the space of all bounded real (or complex) sequences $x = (\xi_j)_{j\geq 1}$ with norm $\|x\|_\infty = \sup_{j\geq 1} |\xi_j|$. Define a linear operator $T : \ell^\infty \to \ell^\infty$ by $y = (\eta_j) = Tx$, $\eta_j = \frac{\xi j}{j} (j \geq 1)$. Show that T is linear and bounded on ℓ^∞, and determine its operator norm $\|T\|$. 5

4. (a) Prove that any two norms on a finite-dimensional vector space are equivalent. 4

(b) Let $C[-1,1]$ be the space of continuous functions on $[-1,1]$. Define a linear functional $f: C[-1,1] \to \mathbb{R}$ by $f(x) = \int_{-1}^{0} t x(t)\,dt - \int_{0}^{1} x(t)\,dt$. Find the operator norm $\|f\|$. 5

(c) In $\mathbb{R}^3$, let $e_1 = (1,1,1)$, $e_2 = (1,1,-1)$, $e_3 = (1,-1,-1)$, let $\{f_1, f_2, f_3\} \subset (\mathbb{R}^3)^*$ be the dual basis to $\{e_1, e_2, e_3\}$. For $x = (1, 0, 0)$, compute $f_1(x), f_2(x), f_3(x)$. 5

UNIT–III

5. (a) Let X be a normed space and Y a Banach space. Denote by $\mathcal{B}(X,Y) = \{T : X \to Y \mid T \text{ linear and bounded}\}$ the space of bounded linear operators with the operator norm $\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$. Prove that $\mathcal{B}(X,Y)$ is complete. 7

(b) Show that the projection $T: \mathbb{R}^2 \to \mathbb{R}, T(x_1, x_2) = x_1$, is an open map. Is the map $S: \mathbb{R}^2 \to \mathbb{R}^2, S(x_1, x_2) = (x_1, 0)$, an open map? Give an example to show that an open mapping need not send closed sets to closed sets. 7

6. (a) Show that the dual space of ℓ^1 is ℓ^∞. 7

(b) Let X and Y be normed spaces with X compact. Suppose $T: X \to Y$ is a bijective closed linear operator. Prove that the inverse $T^{-1}: Y \to X$ is bounded. 7

UNIT–IV

7. (a) On $\mathbb{R}^n$, consider the standard inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, $x, y \in \mathbb{R}^n$. Prove that $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Give an example of a normed space that is not a Hilbert space and justify why. 7

(b) Prove Bessel's inequality and deduce Parseval's identity for a complete orthonormal set. 7

8. (a) Let X be an inner product space and $M \subset X$. 7

(i) Show that if M is total in X, then there is no nonzero $x \in X$ with $x \perp M$.

(ii) If X is a Hilbert space, prove the converse: if the only $x \in X$ with $x \perp M$ is $x = 0$, then M is total in X.

(b) Let H be a Hilbert space and $\{e_k\}_{k \in I}$ an orthonormal family. Show that $\{e_k\}_{k \in I}$ is total in H if and only if for every $x, y \in H$, $\langle x, y \rangle = \sum_{k \in I} \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$. 7

UNIT–V

9. (a) Let H be a Hilbert space. Prove that for every bounded linear functional f on H there exists a unique $z \in H$ such that $f(x) = \langle x, z \rangle$ for all $x \in H$, and that $\|z\| = \|f\|$. 7

(b) Let H be a Hilbert space and $T : H \to H$ a bijective with bounded linear operator whose inverse is bounded. Show that the $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$. 3

(c) Let H be a complex Hilbert space and let $T_1, T_2 : H \to H$ be bounded linear operators. If $\langle T_1 x, x \rangle = \langle T_2 x, x \rangle$ for all $x \in H$, prove that $T_1 = T_2$. 4

10. (a) Let H be a Hilbert space and let $U, V : H \to H$ be unitary operators. Prove: 9

(i) U is isometric: $\|Ux\| = \|x\|$ for all $x \in H$

(ii) $\|U\| = 1$ (assuming $H \neq \{0\}$)

(iii) $U^{-1} = U^*$ is unitary

(iv) UV is unitary

(v) U is normal $(UU^* = U^*U)$

(b) Let X be a finite-dimensional inner product space and $T : X \to X$ a linear operator. If T is self-adjoint, show that its spectrum is real. If T is unitary, show that its eigenvalues have absolute value 1. 5