November 2025
M.Sc.
First Semester
CORE – 04
MATHEMATICS
Course Code: MMAC 1.41
(Abstract Algebra)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Prove that the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ form a multiplicative group. 4

   (b) Let G be a group such that $a^2 = e \forall a \in G$. Show that G is abelian. Is the same true if $a^3 = e \forall a \in G$. 5

   (c) If H and K are cyclic groups of order m and n respectively, such that $(m, n) = 1$. Show that $H \times K$ is a cyclic group of order mn. 5

2. (a) Show that A_4 is the only subgroup of order 12 in S_4. 4

   (b) Prove that $G/_{Ker\, f} \cong \operatorname{Im} f$, where $f : G \to G'$ is a group homomorphism. 5

   (c) Let $\phi : G \to G'$ be an epimorphism of groups. Prove that the following assertions hold: 5

   (i) $H \leq G \Rightarrow \phi(H) \leq G'$

   (ii) $H \trianglelefteq G \Rightarrow \phi(H) \trianglelefteq G'$

   (iii) $H \leq G$ and $Ker\,\phi \subset H \Rightarrow H = \phi^{-1}(\phi(H))$

UNIT–II

3. (a) Define group action and explain group action by conjugation. 4

(b) State and prove Cayley's theorem. 5

(c) Let G be a finite p-group and H a proper subgroup of G. Then show that H is a proper subgroup of normalizer of H in G. 5

4. (a) State and prove Sylow's third theorem. 6

(b) Let G be a finite p-group. Show that for all non-trivial normal subgroups N of G, $N \cap Z(G) \neq \{e\}$. 4

(c) Let G be a group containing an element of finite order $n > 1$ and exactly two conjugacy classes. Prove that $|G| = 2$. 4

UNIT–III

5. (a) Prove that if P is the only Sylow p-subgroups of G, then P is normal in G and conversely. 4

(b) Show that there is no simple group of order 48. 5

(c) Prove that the alternating group A_n is generated by the set of all 3-cycles in S_n. 5

6. (a) If p, q, r, are primes such that $p < q < r$. Prove that there are no simple groups of order $p\ q\ r$. 5

(b) Prove that A_n is simple $\forall n \geq 5$. 5

(c) Find all the non-isomorphic abelian groups of order 360. 4

UNIT–IV

7. (a) If R is a division ring, then prove that $Z(R)$ is a field. 4

(b) If R is an integral domain (with unity), prove the following assertions: 6

(i) A non-zero idempotent cannot be nilpotent

(ii) R contains no idempotents except 0 and unity

(iii) For $a, b \in R$, if $a^m = b^m, a^n = b^n$ and $(m, n) = 1$, then $a = b$.

(c) Let $f : R \to S$ be a ring homomorphism. Show that f is 1-1 if and only if $\ker f = \{0\}$. 4

8. (a) Define ideal of a ring R. Consider the ring of 2×2 matrices over the real $\mathbb{R}$ and let $I = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} : a, b \text{ are reals} \right\}$. Show that I is a left ideal but not a right ideal of R. 5

(b) Let R be a commutative ring with unity and let I be an ideal of R. Prove that I is prime ideal if and only if $\frac{R}{I}$ is an integral domain. 5

(c) If R is a field, then prove that $R[x]$ is PID. 4

UNIT–V

9. (a) Let $f(x), g(x) \in Z[x]$ be two primitive polynomials. Then the product $f(x) \cdot g(x)$ is also primitive. 4

(b) If $p(x)$ is a polynomial over a field F. Then prove that $\frac{F[x]}{\langle P(x) \rangle}$ is a field if and only if $p(x)$ is irreducible over F. 5

(c) Show that the polynomial $x^2 + x + 2$ is irreducible over $F = \{0,1,2\}$ modulo 3. Use it to construct a field of 9 elements. 5

10. (a) Show that there are infinitely many polynomials over the field $\mathbb{Z}_2$. 4

(b) Show that the polynomial $f(x) = 2x^2 + 4$ is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$. 4

(c) Explain Gauss lemma for irreducible of $f(x) \in \mathbb{Z}[x]$. Show that $f(x) = x^3 - 9x^2 + 15x - 2$ is irreducible over neither $\mathbb{Q}$ nor $\mathbb{Z}$. 6