November 2025
M.Sc.
First Semester
CORE – 03
MATHEMATICS
Course Code: MMAC 1.31
(Real Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Let z and w be complex numbers. Show that: 2×2=4

   (i) $z\bar{z}$ is real and non-negative

   (ii) $|z+w| \le |z| + |w|$

   (b) Prove that closed subsets of compact sets are compact. 5

   (c) Prove that for any finite collection $G_1, G_2, ..., G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open. Is the result true for an arbitrary intersection? 5

2. (a) Let E° denote the set of all interior points of E. Prove that:

   (i) E° is always open 2×2=4

   (ii) E is open if and only if $E^\circ = E$

   (b) If z is a complex number such that $|z| = 1$, then compute the value of $|1+z^2| + |1-z^2|$. 5

   (c) Define the closure of a set E, where E is a subset of a metric space X. Prove that the closure of E is closed in X. 5

UNIT–II

3. (a) Show that $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$. 4

(b) Suppose $\{s_n\}$ and $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Show that $\lim_{n\to\infty} s_n t_n = st$. 5

(c) Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, $cn = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0,1,2,...)$. Show that $\sum_{n=0}^{\infty} c_n = AB$. 5

4. (a) If $\sum a_n = A$ and $\sum b_n = B$, show that $\sum (a_n + b_n) = A + B$ and $\sum ca_n = cA$, for any fixed complex number *c*. 4

(b) If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed positive integer, and if $\sum c_n$ converges, then show that $\sum a_n$ converges. 5

(c) Show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. 5

UNIT–III

5. (a) Suppose *a* and *c* are real numbers, *c* > 0, and *f* is defined on $[-1,1]$ by $f(x) = \begin{cases} x^a \sin(x^{-c}) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$. Prove that: 2×2=4

(i) *f* is continuous if and only if *a* > 0

(ii) $f'(0)$ exists if and only if *a* > 1

(b) Let *X* and *Y* be metric spaces; suppose *f* maps *E* into *Y*, where $E \subset X$ and *p* is a limit point of *E*. Prove that $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in *E* such that $p_n \ne p$ and $\lim_{n\to\infty} p_n = p$. 5

(c) Prove that continuous mapping of connected subsets are connected. 5

6. (a) Let *f* be defined on $[a,b]$; if *f* has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then show that $f'(x) = 0$. 4

(b) Show that continuous mapping of a compact space is compact. 5

(c) Suppose f is defined in a neighbourhood of x and suppose $f''(x)$ exists. Show that
$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$. Show by an example that the limit may exist even if $f''(x)$ does not. 5

UNIT–IV

7. (a) Suppose $f \geq 0$ is continuous on $[a,b]$, and $\int_a^b f(x)dx=0$. Prove that $f(x)=0$ for all $x\in[a,b]$. 4

(b) Show that $f\in\mathcal{R}(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon>0$ there exists a partition P such that $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$. 5

(c) State and prove the fundamental theorem of calculus. 5

8. (a) If $f(x)=0$ for all irrational x, $f(x)=1$ for all rational x, prove that $f\notin\mathcal{R}$ on $[a,b]$ for any $a<b$. 4

(b) If $f\in\mathcal{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then show that $f\in\mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and
$\int_a^c f\,d\alpha+\int_c^b f\,d\alpha=\int_a^b f\,d\alpha$. 5

(c) If P^* is a refinement of P, then show that $L(P,f,\alpha)\leq L(P^*,f,\alpha)$ and $U(P^*,f,\alpha)\leq U(P,f,\alpha)$. 5

UNIT–V

9. (a) Suppose $\lim_{n\to\infty} f_n(x)=f(x)\,(x\in E)$ and $M_n=\sup_{x\in E}|f_n(x)-f(x)|$. Prove that $f_n\to f$ uniformly on E if and only if $M_n\to 0$ as $n\to\infty$. 4

(b) Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Prove that $\mathcal{B}$ is a uniformly closed algebra. 5

(c) Construct sequences $\{f_n\}$ and $\{g_n\}$ which converges uniformly on the set E, but $\{f_n g_n\}$ does not converge uniformly on E. 5

10. (a) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval. 4

(b) Prove that the sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N, n \geq N, x \in E$ implies $\left|f_n(x) - f_m(x)\right| \leq \varepsilon$. 5

(c) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for n = 1, 2, 3, …, and suppose $f_n \to f$ uniformly on $[a, b]$. Prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$. 5