

November 2025
M.Sc.
First Semester
CORE – 02
MATHEMATICS
Course Code: MMAC 1.21
(Linear Algebra)

Total Mark: 70

Pass Mark: 28

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT-I

1. (a) Prove that the sum of subspaces $U_1, \dots, U_m \subseteq V$ is a subspace and is the smallest subspace containing all U_j . 4
 - (b) Let U be the subspace of \mathbb{C}^5 defined by
$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \mid 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}$$
. 5
 - (i) Find a basis of U .
 - (ii) Extend the basis obtained in part (a) to a basis of \mathbb{C}^5 .
 - (iii) Find a subspace W of \mathbb{C}^5 such that $\mathbb{C}^5 = U \oplus W$.
 - (c) Let U_1, \dots, U_m be finite-dimensional subspaces of a vector space V such that $U_1 + \dots + U_m$ is a direct sum. Prove that $U_1 \oplus \dots \oplus U_m$ is finite-dimensional and that
$$\dim(U_1 \oplus \dots \oplus U_m) = \dim U_1 + \dots + \dim U_m$$
. 5
2. (a) Let V be a vector space over a field \mathbb{F} and let $v_1, \dots, v_m \in V$ be linearly independent. For $w \in V$, prove that the set $\{v_1, \dots, v_m, w\}$ is linearly independent if and only if $w \notin \text{span}(v_1, \dots, v_m)$. 5
 - (b) Let V and W be finite-dimensional vector spaces over the same field \mathbb{F} . Prove that $V \cong W \Leftrightarrow \dim V = \dim W$. 5
 - (c) If S, T are invertible linear maps, show $(ST)^{-1} = T^{-1}S^{-1}$. 4

UNIT-II

3. (a) Let V be a finite-dimensional vector space over a field \mathbb{F} , let $T \in \mathcal{L}(V)$, and let $\lambda \in \mathbb{F}$. Prove that the following are equivalent: 1×4=4
- (i) λ is an eigenvalue of T
 - (ii) $T - \lambda I$ is not injective
 - (iii) $T - \lambda I$ is not surjective
 - (iv) $T - \lambda I$ is not invertible
- (b) Over \mathbb{C} , prove that every operator $T \in \mathcal{L}(V)$ has an eigenvalue. 5
- (c) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. 5
- (i) Suppose T is diagonalizable. Prove that $V = \text{null}T \oplus \text{range}T$.
 - (ii) Prove the converse of (i) or give a counterexample to the converse.
4. (a) Let V be a finite-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Prove that there exists a basis of V with respect to which the matrix of T is upper triangular. 5
- (b) Let $T \in \mathcal{L}(V)$ and let $v \neq 0$ be an eigenvector of T with eigenvalue $\lambda \in \mathbb{F}$, i.e., $Tv = \lambda v$. Let $p \in P(\mathbb{F})$ be a polynomial. Prove that $p(T)v = p(\lambda)v$. 4
- (c) Let V be a finite-dimensional complex vector space and let $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$ for every $\lambda \in \mathbb{C}$. 5

UNIT-III

5. (a) Let $T \in \mathcal{L}(V)$ with $n = \dim V$. Prove that the sequence of null spaces stabilizes at step n , i.e.
- $$\text{null}(T^n) = \text{null}(T^{n+1}) = \text{null}(T^{n+2}) = \dots$$
- 4

- (b) Let $T \in \mathcal{L}(V)$, let $m \in \mathbb{N}$, and suppose $v \in V$ satisfies $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that the list $v, Tv, T^2v, \dots, T^{m-1}v$ is linearly independent. 5
- (c) Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (4z_2, 0, 5z_3)$. 5
- (i) Find all eigenvalues of T , the corresponding eigenspaces, and the corresponding generalized eigenspaces.
- (ii) Show that \mathbb{C}^3 is the direct sum of the generalized eigenspaces corresponding to the distinct eigenvalues of T .
6. (a) Let $N \in \mathcal{L}(\mathbb{F}^4)$ be the nilpotent operator defined by $N(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3)$. Find the characteristic polynomial and the minimal polynomial of N , and justify your answers. 5
- (b) Let V be a complex vector space and $T \in \mathcal{L}(V)$. Let q be the characteristic polynomial of T . Prove that $q(T) = 0$. 5
- (c) Let $N: \mathbb{F}^3 \rightarrow \mathbb{F}^3$ be nilpotent with $N^3 = 0$ and $N^2 \neq 0$. Determine the Jordan form of N and exhibit one Jordan basis. 4

UNIT-IV

7. (a) Let $p > 0$. Prove that there exists an inner product on \mathbb{R}^2 whose induced norm satisfies $\|(x, y)\| = (|x|^p + |y|^p)^{1/p}$ for all $(x, y) \in \mathbb{R}^2$ if and only if $p = 2$. 4
- (b) Let V be a finite-dimensional inner product space and let $\varphi: V \rightarrow \mathbb{F}$ be a linear functional. Prove that there exists a unique vector $u \in V$ such that $\varphi(v) = \langle v, u \rangle$ for all $v \in V$. 5
- (c) Let $C([-1, 1])$ be the vector space of continuous real-valued functions on $[-1, 1]$ with inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ for $f, g \in C([-1, 1])$. Define the linear functional $\varphi: C([-1, 1]) \rightarrow \mathbb{R}$ by $\varphi(f) = f(0)$. Show that there does not

exist $g \in C([-1,1])$ such that $\varphi(f) = \langle f, g \rangle$ for every $f \in C([-1,1])$. 5

8. (a) Let V be a finite-dimensional inner product space and let $U \subseteq V$ be a subspace. Prove that $V = U \oplus U^\perp$. 5

(b) Let V be a complex inner product space and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$. 5

(c) Make $P_2(\mathbb{R})$ into an inner product space by $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Define $T \in \mathcal{L}(P_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$. Show that T is not self-adjoint with respect to this inner product. 4

UNIT-V

9. (a) For a bilinear form $B: V \times V \rightarrow \mathbb{F}$ and $x, y \in V$, define $x_L(t) = B(x, t)$, $y_R(s) = B(s, y)$. 4

(i) Show $L: x \mapsto x_L$ and $R: y \mapsto y_R$ are linear maps $V \rightarrow V^*$.

(ii) Define $V^{\perp_L} = \ker L$ and $V^{\perp_R} = \ker R$. If B is symmetric, Prove $V^{\perp_L} = V^{\perp_R}$.

(b) Let B be a bilinear form on a finite-dimensional V with matrix (b_{ij}) in some basis. Prove the equivalences: 5

(i) $V^{\perp_L} = \{0\}$

(ii) $V^{\perp_R} = \{0\}$

(iii) (b_{ij}) is invertible

(c) Assume B is a symmetric bilinear form on V . Show that the induced form $\bar{B}: V/V^\perp \times V/V^\perp \rightarrow \mathbb{F}$, $\bar{B}(x+V^\perp, y+V^\perp) = B(x, y)$, is well defined, symmetric, and nondegenerate. 5

10. (a) (i) If B is a symmetric bilinear form on V , show
 $Q(x) := B(x, x)$ is a quadratic form.
- (ii) Let $A = (a_{ij})$ be the matrix of a symmetric bilinear form B
in a basis $\{e_i\}$. Express $Q(x) = B(x, x)$ in coordinates. 4
- (b) (i) Define a positive definite Hermitian form. Explain why
every positive definite Hermitian form is an inner product.
- (ii) Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}$, $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and
define $H(x, y) = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$. Show that H is a Hermitian
form on F^n and that it is positive definite. 5
- (c) Let H be nondegenerate. Show that the set of H -unitary
operators on V forms a group under composition (the H -unitary
group). Furthermore, prove that for any H -unitary operator T ,
 $T^{-1} = T^*$, where T^* is the adjoint with respect to H . 5
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