November 2025
M.Sc.
First Semester
CORE – 01
MATHEMATICS
Course Code: MMAC 1.11
(Ordinary Differential Equations)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Show that every solution of the constant coefficient equation $L(y) = y'' + a_1 y' + a_2 y = 0$ tends to zero as $x \to \infty$ if and only if, the real parts of the roots of the characteristic polynomial are negative. 5

   (b) Explain the method of variation of parameters to obtain the solution of the second order non-homogeneous differential equation $y'' + a_1 y' + a_2 y = b(x)$. 5

   (c) Prove that if the Wronskian of solutions of a second-order linear homogeneous differential equation is non-zero at any point in an interval, then the solutions are linearly independent on that interval. 4

2. (a) State and explain the existence theorem for initial value problem of second-order linear differential equations. 5

   (b) Let ϕ be a solution of the differential equation $y'' + a_1 y' + a_2 y = 0$, where a_1, a_2 are constants. If $\psi(x) = e^{(a_1/2)x}\phi(x)$, then show that ψ satisfies the equation $y'' + ky = 0$ for some constant *k*. Compute *k*. 5

   (c) State and prove a formula for the Wronskian of solutions to a second-order linear homogeneous differential equation. 4

UNIT–II

3. (a) Consider a linear homogeneous differential equation of n^{th} order with variable coefficients of the form
$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0$, where
$a_0, a_1, \ldots a_n$ are complex valued functions on some interval *I*.
Show that there exists *n* linearly independent solutions of
$L(y) = 0$ on *I*. 5

(b) Consider the equation $y'' + \alpha(x)y = 0$ where α is a continuous function on $x \in \mathbb{R}$ which is of period $\xi > 0$. Let ϕ_1 and ϕ_2 be the basis for the solutions satisfying $\phi_1(0) = 1, \phi_2(0) = 0$, $\phi_1'(0) = 0, \phi_2'(0) = 1$. Show that there is at least one non-trivial solution ϕ of period ξ if and only if $\phi_1(\xi) + \phi_2'(\xi) = 2$. 5

(c) Consider the second-order linear differential equation
$x^2 y'' + xy' - y = 0,\ x > 0$. 4

(i) Show that there is a solution of the form x^r for any constant *r*.

(ii) Find the solutions ϕ_1 and ϕ_2 satisfying conditions

$\phi_1(1) = 1,\ \phi_1'(1) = 0$ and $\phi_2(1) = 0,\ \phi_2'(1) = 1$.

4. (a) Consider the equation $y'' + \alpha(x)y = 0$, where α is a continuous function on $0 < x < \infty$. If $\alpha(x) \geq \xi$ for $0 < x < \infty$, where ξ is a positive constant then show that every real valued solution has an infinity of zeros on $0 < x < \infty$. 5

(b) Let $b_1, b_2 \ldots b_n$ be non-negative constants, such that
$\forall x \in I,\ |a_j(x)| \leq b_j,\ j = 1, 2, 3 \ldots n$ and define *k* by
$k = 1 + b_1 + b_2 + \ldots + b_n$. If x_0 is a point in *I* and ϕ is a solution of the equation $L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0$,
then prove that $\|\phi(x_0)\| e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k(x-x_0)}$, $\forall x \in I$.
5

(c) The equation $y'+a(x)y=0$ has a solution of the form $\phi(x)e^{-\int_{x_0}^{x} a(t)dt}$. Find a solution of $L(y)=y''+a_1(x)y'+a_2(x)y=0$ of the form $\phi(x)e^{\int_{x_0}^{x} p(t)dt}$ where *p* is a function to be determined. Show that ϕ is a solution of $L(y)=0$ if and only if *p* satisfies the first order non-linear equation $y'=-y^2-a_1(x)y-a_2(x)$. 4

UNIT–III

5. (a) One solution of $x^2y''-xy'+y=0,\ (x>0)$ is $\phi_1(x)=x$. Find the solution $\psi(x)$ of $x^2y''-xy'+y=x^2$ satisfying $\psi(1)=1$, $\psi'(1)=0$. 6

(b) Find the Legendre polynomial of degree *n* in descending power of *x*. 8

6. (a) Find two linearly independent power series solutions of the equation $y''-xy'+y=0$. 6

(b) Show that $\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, \text{ if } m\neq n \\ \dfrac{2}{2n+1}, \text{ if } m=n \end{cases}$ where $P_m(x)$ and $P_n(x)$ are Legendre polynomials. 8

UNIT–IV

7. (a) Find all real-valued solutions of $y'=\dfrac{y^2}{xy+x^2}$. 5

(b) Find an integrating factor for the equation $\left(5x^3y^2+2y\right)dx+\left(3x^4y+2x\right)dy=0$ to make it exact and then solve. 5

(c) Consider the initial value problem: $y' = xy + 1, y(0) = 1$. Compute the first four approximations to the solution. 4

8\. (a) Consider the initial value problem: $y' = 1 + y^2$, $y(0) = 0$. Show that all successive approximations $\phi_0, \phi_1, \phi_2, \phi_3, \ldots$ exist for all real x and $\phi_k(x) \to \phi(x)$ for each x satisfying $|x| \le \frac{1}{2}$. 5

(b) Consider the problem $y' = y + \lambda x^2 \sin y$, $y(0) = 1$ where λ is some real parameter and $|\lambda| \le 1$: 2+3=5

(i) Show that the solution ψ of this problem exists for $|x| \le 1$

(ii) Prove that $\left|\psi(x) - e^x\right| \le |\lambda|\left(e^{|x|} - 1\right)$ for $|x| \le 1$.

(c) Consider the initial value problem: $y' = y^2$, $y(0) = 1$. Does the function $f(x, y) = y^2$ satisfy the Lipschitz condition in respect of y in any interval? Justify your answer. 4

UNIT–V

9\. (a) Define a regular singular point of a differential equation. Explain the difference between a regular singular point and an irregular singular point with examples. 4

(b) Find for all solutions $x^2 y'' + 2xy' - 6y = 0$ for $x > 0$. 4

(c) For the equation $9x(1-x)y'' - 12y' + 4y = 0$ identify the type of singular point at $x = 0$ and solve the equation by using the Frobenius method. 6

10\. (a) Show that ± 1 and infinity are regular singular points for the Legendre equation $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$. Find the indicial polynomial and its roots corresponding to regular singular point $x = 1$. 7

(b) Obtain the solution of the Bessel equation:
$x^2 y'' + xy' + (x^2 - n^2)y = 0$ 7