**October 2025**
**B.A./B.Sc.**
**Third Semester**
MINOR – 3
**MATHEMATICS**
*Course Code: MAN 3.11*
(Group Theory)

*Total Mark: 75* *Pass Mark: 30*
*Time: 3 hours*

*Answer* five *questions, taking* at least one *from each unit.*

## UNIT–I

1. (a) Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all *a* and *b* in *G*. 5

   (b) If *a* is an element of a group *G*, with identity element *e*, and if $m,\ n \in \mathbb{Z}$, prove that $a^{m+n} = a^m \cdot a^n$. 5

   (c) Prove that the set of all $2 \times 2$ matrices with entries from $\mathbb{R}$ and determinant +1 is a group under matrix multiplication. Is this group abelian? 5

2. (a) Prove that the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ of four matrices forms a finite multiplicative abelian group. 5

   (b) Show that the set of all complex numbers *z* such that $|z| = 1$, forms a group with respect to multiplication of complex numbers. 5

   (c) If *G* is a group, then show that: 5

   (i) The identity element of *G* is unique

   (ii) For all $a, b \in G,\ (ab)^{-1} = b^{-1}a^{-1}$

## UNIT–II

3. (a) If $H$ is a subgroup of a group $G$, define the centralizer $C(H)$ of $H$. Prove that $C(H)$ is a subgroup of $G$. 5

   (b) If $H$ and $K$ are subgroups of $G$, show that $H \cap K$ is a subgroup of $G$. 5

   (c) Show that every subgroup of a cyclic group is cyclic. 5

4. (a) Define the center of a group. Show that the center of a group $G$ is a subgroup. 5

   (b) Let $a$ be an element of order $n$ in a group and let $k$ be a positive integer. Prove that $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and order of $a^k$ equals $n/\gcd(n,k)$. 5

   (c) Prove that every group of prime order is cyclic. 5

## UNIT–III

5. (a) Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 5

   (b) If $N$ is a normal subgroup of $G$ and $H$ is any subgroup of $G$, prove that $NH$ is a subgroup of $G$. 5

   (c) Let $H$ and $K$ be subgroups of a finite group $G$ with $H \subseteq K \subseteq G$. Prove that $|G:H| = |G:K||K:H|$. 5

6. (a) State and prove the Lagrange's theorem for finite groups. 5

   (b) Let $H$ be a subgroup of $G$, and let $a$ and $b$ belong to $G$. Prove that: 5

      (i) $aH = H$ if and only if $a \in H$

      (ii) $aH = bH$ if and only if $a \in bH$

   (c) Determine whether the following permutations are even or odd:

      (i) $(1\ 2\ 4\ 3)(3\ 5\ 2\ 1)$ 5

      (ii) $(1\ 2)(1\ 3\ 4)(1\ 5\ 2)$

## UNIT–IV

7. (a) State and prove the first isomorphism theorem. 5

(b) Prove that a homomorphism $\phi$ of $G$ into $\overline{G}$ with kernel $K$ is an isomorphism if and only if $K = \{e\}$. 5

(c) Suppose that $\phi$ is an isomorphism from a group $G$ onto a group $\overline{G}$. Prove that: 5

(i) $\phi$ maps the identity of $G$ to the identity of $\overline{G}$.

(ii) $|a| = |\phi(a)|$, where $|x|$ denote the order of $x$.

8. (a) Prove that every group is isomorphic to a group of permutations. 5

(b) Suppose that $\phi$ is an isomorphism from a group $G$ onto a group $\overline{G}$. Prove that, if $K$ is a subgroup of $G$, then $\phi(K) = \{\phi(k) \mid k \in K\}$ is a subgroup of $\overline{G}$. 5

(c) If $\phi$ is a homomorphism of $G$ into $\overline{G}$ with kernel $K$, then show that $K$ is a normal subgroup of $G$. 5