October 2025
B.A./B.Sc.
Fifth Semester
CORE – 12
MATHEMATICS
Course Code: MAC 5.21
(Group Theory - II)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Define the term automorphism of a group G. Find all automorphisms on an infinite cyclic group. 5
   (b) Define the center of a group. Prove that the center of a group is characteristic. 5
   (c) Find two groups G and H such that $G \not\cong H$,but $A(G) \cong A(H)$. 4

2. (a) Define the term inner automorphism of a group G. Show that the inner automorphism of a group G is trivial if and only if G is abelian. 5
   (b) Prove that the characteristic subgroup of a group G is normal in G. Is the converse true? Justify. 5
   (c) Find all automorphisms of $\mathbb{Z}_8$. Show that $A(\mathbb{Z}_8) \cong U(8)$. 4

UNIT–II

3. (a) Let G_1 and G_2 be finite cyclic groups. Show that $G_1 \times G_2$ cyclic if and only if $|G_1|$ and $|G_2|$ are relatively prime, where $|G|$ means the order of G. 5
   (b) Prove that every group of order p^2 , where p is a prime, is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. 5
   (c) Determine the number of elements of order 15 in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$. 4

4. (a) Let G be a finite abelian group of order $p^n m$, where p is a prime that does not divide m. Prove that $G = H \times K$, where $H = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$. Also, show that $\circ(H) = p^n$. 5

(b) Let G = {1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 109,116, 118, 127, 134} under multiplication modulo 135. Express G as an external and an internal direct product of cyclic groups. 5

(c) Prove or disprove that $\mathbb{Z} \times \mathbb{Z}$ is a cyclic group. 4

UNIT–III

5. (a) If a group G act on a non-empty set A, show that the orbits in A under G form a partition of A. 5

(b) Let G be a group, let H be a subgroup of G, and let G act by left multiplication on the set A of left cosets of H in G. Let be the associated permutation representation induced by this action. Prove that the kernel of π_H is $\bigcap_{x \in G} xHx^{-1}$, and that, it is the largest normal subgroup of G contained in H. 5

(c) Let G be a group and $A = G$. Define a map from $G \times A$ to A by $g \cdot a = ag^{-1}$ for $g,\ a \in G$. Does this map define a group action? Justify. 4

6. (a) If G is a finite group of order n and p is the smallest prime dividing n, then prove that any subgroup of index p is normal. 5

(b) Let G be a group acting on a nonempty set A. Define the centralizer of A in G. Show that the centralizer of A in G is a subgroup of G. Compute the centralizer of j in Q_8. 5

(c) Let $G = D_4$ and let $H = \langle f \rangle$, where $f \in D_4$ stands for a reflection. With suitable labelling, label the distinct left cosets of H in G. Under this labelling, compute σ_r induced by the action of left multiplication by the group element r, rotation by 90°, on the left cosets. 4

UNIT–IV

7. (a) Prove that the number of conjugates of a subset S in a group G is the index of the normalizer of S. Moreover, show that the number of conjugates of an element $s \in G$ is the index of the centralizer of s. 5

   (b) Prove that two elements of S_n are conjugates if and only if they have the same cycle type. 5

   (c) Determine the class equation for non-abelian groups of orders 39 and 55. 4

8. (a) Let G be a group with order p^2, where p is a prime, then show that G is Abelian. Further, show that G is isomorphic to either $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. 5

   (b) Find all conjugacy classes and their sizes in D_4. 5

   (c) Let σ be the 5-cycle (1 2 3 4 5) in S_5. Find an explicit element $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^{-2}$. 4

UNIT–V

9. (a) If G is a group of order pq, where p and q are primes, $p < q$, and p does not divide $q - 1$, then prove that G is cyclic. 5

   (b) Let G be a non-cyclic group of order 21. How many Sylow 3-subgroups does G have? 5

   (c) Show that the centre of a group of order 60 cannot have order 4. 4

10. (a) Show that every group of order 56 has a proper nontrivial normal subgroup. 5

    (b) Prove that a group of order 175 is abelian. 5

    (c) Find all Sylow 3-subgroups of S_4. 4
