**2023**

**M.Sc.**

**Third Semester**

DISCIPLINE SPECIFIC ELECTIVE – 02

**MATHEMATICS**

*Course Code: MMAD 3.21*

(Tensor Analysis & Riemannian Geometry)

*Total Mark: 70* *Pass Mark: 28*

*Time: 3 hours*

*Answer five questions, taking one from each unit.*

## UNIT–I

1. (a) Prove that the transformation of contravariant vector, covariant vector and mixed tensors posses the group property. 6

(b) State and prove the quotient law of tensors. 4

(c) If $A^{ij}$ is contravariant tensor and $B_i$ is covariant vector then show that $A^{ij}B_k$ is a tensor of rank three, but $A^{ij}B_j$ is a tensor of rank one. 4

2. (a) Define symmetric and anti symmetric tensors, find the number of independent components of these two tensors and prove that for symmetric tensor symmetric property remains unchanged by tensor law of transformation. 5

(b) Show that Kronecker delta is a mixed tensor of rank two by using quotient law of tensors also show that it is invariant. Prove that if the components of a tensor vanishes in one coordinate system, they vanish identically in all coordinate system. 4

(c) Show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to transformation of the type $x'^{a} = a^{\alpha}_{\beta} x^{\beta} + b^{\alpha}$ where $a's$ and $b's$ are constants such that

$$\sum_{\alpha=1}^{3} a^{\alpha}_{\beta} a^{\alpha}_{\gamma} = \delta^{\alpha}_{\gamma}.$$ 3

(d) Prove that $A_{ij}B^iC^j$ is invariant if $B^i$ and $C^j$ are vectors and $A_{ij}$ is a tensor. 2

## UNIT–II

3. (a) Show that: 5

   (i) $g_{ij}$ is a second rank covariant tensor and

   (ii) $g_{ij}dx^i dx^j$ is an invariant

   (b) Show that the angle between the contravariant vectors is real when the Riemannian metric is positive definite. And find the condition that two vectors $A^i$ and $B^j$ be orthogonal. 4

   (c) Find the metric of a Euclidean space referred to spherical co-ordinates. 5

4. (a) Define $n$-ply of orthogonal system of hypersurfaces and prove the necessary and sufficient condition for the existence of an $n$-ply orthogonal system of co-ordinate hypersurfaces is that fundamental form must be of the form $ds^2 = \sum_{i=1}^{n} g_{ij}\left(dx^i\right)^2$. 5

   (b) Prove that the inclination θ of two vectors has the same value whether they are regarded as vectors in $V_n$, or as vectors in Euclidean space $S_m$ in which $V_n$ is immersed. 4

   (c) Discuss in detail the principal directions for a symmetric covariant tensor of second kind. 5

## UNIT–III

5. (a) Show that Christoffel's symbols are not tensor quantities. 5

   (b) Obtain the covariant derivative of contravariant vector. 4

   (c) Show that metric tensors are covariant constant with respect to Christoffel's symbols. 5

6. (a) Prove that the laws of transformations of Christoffel's symbols possess the group properties. 4

   (b) Obtain the covariant derivative of $A^{ij}$. 6

   (c) Show that if $A^{ij}$ is a symmetric tensor, then

$$A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) - \frac{1}{2}A^{jk}\frac{\partial g_{jk}}{\partial x^i}.$$ 4

## UNIT–IV

7. (a) Define geodesic and find the differential equation of it in a $V_n$, using the property that it is a path of maximum (or minimum) length joining two points on it. 5

(b) Show that it is always possible to choose a geodesic co-ordinate system of any $V_n$ with an arbitrary pole $P_0$. And prove the necessary and sufficient condition that the hypersurfaces $\phi$ = constant form a system of parallels is that $(\nabla\phi)^2 = 1$. 3+3=6

(c) Prove that every Riemannian co-ordinate system is necessarily a geodesic co-ordinate system, but the converse is not true. 3

8. (a) Discuss the Levi-Cita's concept of parallelism of vectors. 5

(b) If two vectors of constant magnitudes undergo parallel displacements along a given curve, then show that they are inclined at a constant angle. Also, prove that geodesics are the auto-parallel curve. 3+1=4

(c) State and prove the fundamental theorem of Riemannian geometry. 5

## UNIT–V

9. (a) Obtain an expression for Riemannian-Christoffel tensor of second kind and show that it can contracted in two ways-one of these leads to a zero tensor and other to a symmetric tensor. 4+4=8

(b) Prove the necessary and sufficient condition that the congruence $e_{n|}$ of an orthonormal ennuple to be normal is that

$$\gamma_{npq} = \gamma_{nqp}\ (p, q = 1, 2, ..., n-1).$$ 6

10. (a) Define curvature of congruence and obtain necessary and sufficient conditions that a congruence be a geodesic congruence. 5

(b) Show the necessary and sufficient condition for a Riemannian $V_n (n > 3)$ to be of constant Riemannian curvature is that the Weyl tensor vanishes identically throughout $V_n$. 6

(c) Prove that if $R_i^a = g^{aj} R_{ij}$, then $R^a_{i,a} = \frac{1}{2}\frac{\partial R}{\partial x^i}$ and deduce that when $n > 2$ the scalar curvature of an Einstein space is constant. 3

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