2023
M.Sc.
First Semester
CORE – 04
MATHEMATICS
Course Code: MMAC 1.41
(Abstract Algebra)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Prove that in any group, an element and its inverse have the same order. 5
(b) Prove that the union of two subgroups of a group G is a subgroup of G if and only if one is contained in the other. 5
(c) Prove that the order of a cyclic group is equal to the order of its generator. 4

2. (a) State and prove the fundamental theorem of group homomorphism. 5
(b) Prove that a subgroup N of a group G is a normal subgroup of G if and only if every left coset of N in G is also a right coset of N in G. 5
(c) Show that every permutation of a finite set can be expressed as a product of transpositions. 4

UNIT–II

3. (a) Explain group action by conjugation. 4
(b) State and prove Cayley's theorem. 6
(c) If G is a finite group, show that G is a p-group if and only if order of G is p^n. 4

4. (a) State and prove Sylow's first theorem. 7
(b) Prove that the dihedral group D_n has $2n$ elements. 7

UNIT–III

5. (a) Show that no group of order 30 is simple. 4

(b) Show that the relation of conjugacy is an equivalence relation. 5

(c) If G_1 and G_2 are two groups, show that $G_1 \times G_2$ is abelian if and only if both G_1 and G_2 are abelian. 5

6. (a) Find all the conjugate classes in S_4 and verify the class equation. 7

(b) How many abelian groups (up to isomorphism) are there of order 15? 4

(c) What is the smallest positive integer n such that there are two nonisomorphic groups of order n? Name the two groups. 2+1=3

UNIT–IV

7. (a) Define characteristic of a ring and show that the characteristic of an integral domain is either zero or a prime. 5

(b) Prove that the sum of two ideals of a ring R is also an ideal of R containing both the ideals. 6

(c) Prove that the only homomorphisms from the ring of integers $\mathbb{Z}$ to $\mathbb{Z}$ are the identity and the zero mappings. 3

8. (a) Show that $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field. 4

(b) Prove that every Euclidean domain is a PID. 5

(c) Show that in a UFD an element is prime if and only if it is irreducible. 5

UNIT–V

9. (a) If F is a field, prove that $F[x]$ is a PID. 5

(b) Prove that any polynomial which is reducible over rational numbers, is reducible over integers. 5

(c) State and prove the factor theorem. 4

10. (a) State and prove Eisenstein's criterion for irreducibility. 5

(b) Show that the polynomial $x^2 + x + 2$ is irreducible in $\mathbb{Z}_3[x]$ and use it to construct a field of 9 elements. 5

(c) Show that the polynomial $x^{p-1} + x^{p-2} + \ldots + x + 1$ is irreducible over $\mathbb{Q}$. 4
