2023
M.Sc.
First Semester
CORE – 03
MATHEMATICS
Course Code: MMAC 1.31
(Real Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) For $x, y \in \mathbb{R}$, does $d(x, y) = \frac{|x-y|}{1+|x-y|}$ define a metric? Justify. 4

   (b) Prove that $\#]0,1[= c$, where c is the cardinality of $\mathbb{R}$. 5

   (c) Is $\mathbb{R}$ compact? Justify. 5

2. (a) Prove that the union of two countable sets is countable. 4
   (b) Prove/disprove: Every subset of a discrete metric space is open. 4
   (c) Prove that the cardinality of $\mathbb{R}$ is equal to $2^{\aleph_0}$, where $\aleph_0$ is the cardinality of a countable set. 6

UNIT–II

3. (a) Prove that a series of nonnegative terms converges if and only if the sequence of its partial sums is bounded. 6
   (b) Let $a_1 \geq a_2 \geq \ldots \geq 0$. Prove that the series $\sum a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. 8

4. (a) Prove/disprove: The limit of a convergent sequence (x_n) is same as the limit point of its range. 3

(b) Prove that e is irrational. 5

(c) Show that the product of two convergent series may diverge. 6

UNIT–III

5. (a) For metric spaces X and Y, prove that a function f is continuous from X to Y if and only if for every open set E in Y, $f^{\leftarrow}(E)$ is open in X. 6

(b) Give an example of a function f which is differentiable at x but f' is not differentiable at x. 4

(c) Show that the mean value theorem is not valid for complex-valued functions in general. 4

6. (a) If X is compact and $f: X \to \mathbb{R}$ is continuous, prove that f is bounded. Further, prove that f attains its infimum and supremum. 2+5=7

(b) Let $f: X \to Y$ be continuous, where X, Y are metric spaces. If X is compact, prove that $f(X)$ is also compact. 7

UNIT–IV

7. (a) $f:[a,b] \to \mathbb{R}$ is bounded and P, P^* are partitions of such that P^* is finer than P. Prove that $U(P^*, f) \le U(P, f)$. 7

(b) If f is continuous on $[a,b] \subset \mathbb{R}$ and f is real-valued, prove that is f is integrable. Give an example to show that the converse need not be true. 4+3=7

8. (a) Prove that $f:[a,b]\to\mathbb{R}$ is integrable if and only if for a given , there exists a , a partition of $[a,b]$ with $\|P\|<\delta$, and a real number L such that for all $t_i\in[x_{i-1},x_i]$, $\left|\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})-L\right|<\varepsilon$. 7

(b) State and prove the fundamental theorem of calculus. 7

UNIT–V

9. (a) Show that the sequence $\left(x^2e^{-nx}\right)$ converges uniformly on $[0,\infty[$. 5

(b) For sequences of functions, describe the difference between pointwise convergence and uniform convergence with examples. 7

(c) Give an example of a sequence of functions which converges pointwise, but not uniformly, to some function. 2

$\varepsilon>0$ $\delta>0$ $P=\{x_0,x_1,...,x_n\}$

10. (a) Let (f_n), (g_n) be sequences of bounded functions on A that converges uniformly on A to f, g, respectively. Show that (f_n+g_n) converges unifromly on A to $f+g$. 7

(b) Show that $\lim\left(x^2e^{-nx}\right)=0$ and that $\lim\left(n^2x^2e^{-nx}\right)=0$ for $x\in\mathbb{R}, x\geq 0$. 7