2023

M.Sc.

First Semester

CORE – 02

MATHEMATICS

Course Code: MMAC 1.21

(Linear Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Let V be a vector space over a field F. Show that a non-empty subset W of V is a subspace of V if and only if, for each pair of vectors $\alpha, \beta \in W$ and each scalar c in F, the vector $c\alpha + \beta$ is again in W. 5

   (b) Show that the span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list. 5

   (c) Suppose that $T \in \mathcal{L}(V, W)$ is injective and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly independent in V. Prove that $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ is linearly independent in W. 4

2. (a) If V is a vector space over a field F, show that the zero vector in V and the additive inverse of a vector in V are both unique. 5

   (b) Let V and W be vector spaces over the field F and let T be a linear transformation form V into W. Suppose V is finite-dimensional, then prove that rank(T) + nullity(T) = dim(V). 5

   (c) Suppose V is a finite-dimensional vector space over the field F and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$. 4

UNIT–II

3. (a) Suppose V is finite dimensional vector space over the field F, $T \in \mathcal{L}(V)$ and $c \in F$. Prove that c is a characteristic value of T if and only if $(T - cI)$ is not invertible. 5

(b) If A is an $n \times n$ matrix, then prove that there exists a non-singular matrix P such that $P^{-1}AP$ is an upper triangular matrix with the characteristic values on the diagonal. 5

(c) Show that matrices $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are similar matrices over $\mathbb{C}$. 4

4. (a) Let V be a finite dimensional vector space over the field F and let $T \in \mathcal{L}(V)$. Prove that T satisfies at least one polynomial of positive degree over F. 5

(b) Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities. 5

(c) Define $T \in \mathcal{L}(\mathbb{R}^3)$ by

$T(x, y, z) = (7x - 4y + 10z, 4x - 3y + 8z, -2x + y - z)$. Find the characteristic values of T and an ordered basis $\mathcal{B}$ for $\mathbb{R}^3$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix. 4

UNIT–III

5. (a) Let T be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & 4 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is a characteristic vector of T. 5

(b) If V is a finite dimensional vector space and $W_1, W_2, \ldots, W_k$ are subspaces of V. Let $W = W_1 + W_2 + \ldots + W_k$, then show that the following are equivalent: 5

(i) $W_1, W_2, \ldots, W_k$ are independent

(ii) For each $j, 2 \le j \le k, W_j \cap (W_1 + \ldots + W_{j-1}) = \{0\}$

(iii) If $\mathcal{B}_i$ is an ordered basis for W_i, $1 \le i \le k$ then the sequence $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ is an ordered basis for W.

(c) Let E be a projection on V and $T \in \mathcal{L}$. Prove that both range and the null space of E are invariant under T if and only if $TE = ET$. 4

6. (a) Let $T \in \mathcal{L}(V)$. Suppose $c_1, c_2, \ldots, c_m$ are distinct characteristic values of T and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are corresponding characteristic vectors. Prove that $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent. 5

(b) If $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, then prove that there exist k linear operators $E_1, E_2, \ldots, E_k$ on V such that 5

(i) each E_i is a projection

(ii) $E_i E_j = 0$ if $i \neq j$

(iii) $I = E_1 + E_2 + \ldots + E_k$

(iv) the range of E_i is W_i

(c) Reduce the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ to Jordan canonical form. 4

UNIT–IV

7. (a) Suppose α and β are two vectors in an inner product space V, then show that $|\langle \alpha, \beta \rangle| \le \|\alpha\| \|\beta\|$. Under what conditions is the inequality an equality? 5

(b) If V is an inner product space, prove that $T \in \mathcal{L}(V)$ is normal if and only if $\|T(\alpha)\| = \|T^*(\alpha)\|$ for all $\alpha \in V$. 5

(c) Suppose $c_1, c_2, \ldots, c_n$ are scalars with absolute value 1 and $S \in \mathcal{L}(V)$ satisfies $S(e_j) = c_j e_j$ for some orthonormal basis $e_1, e_2, \ldots, e_n$ of V. Show that S is an isometry. 4

8. (a) Suppose U is a finite-dimensional subspace of an inner product space V, then prove that $U = (U^{\perp})^{\perp}$. 5

(b) Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is Hermitian. Prove that characteristic vectors of T corresponding to distinct characteristic values are orthogonal. 5

(c) Prove that if V is a complex inner product space, then

$$\langle\alpha,\beta\rangle = \frac{\|\alpha+\beta\|^2 - \|\alpha-\beta\|^2}{4} + \frac{\|\alpha+i\beta\|^2 - \|\alpha-i\beta\|^2}{4}i$$

for all $\alpha,\beta \in V$. 4

UNIT–V

9. (a) Prove that the mapping $B : \mathcal{P}_3(\mathbb{R}) \times \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$ defined by $B(p,q) = \int_0^1 p(x)q(x)dx$ is a bilinear form on $\mathcal{P}_3(V)$. Write the matrix of B with respect to the standard basis of $\mathcal{P}_3(\mathbb{R})$. 5

(b) Let V be a finite-dimensional vector space over a field of characteristic zero, and let B be a symmetric bilinear form on V. Prove that there is an ordered basis for V in which B is represented by a diagonal matrix. 5

(c) Let f be the form on $\mathbb{R}^2$ defined by $f((x_1, y_1),(x_2, y_2)) = x_1y_1 + x_2y_2$. Find the matrix of f with respect to the basis $\{(1, 2),(3, 4)\}$. 4

10. (a) Let V be a vector space over the field F of characteristic $\neq 2$. Then prove that the mapping $f : B \to Q$, where $Q(x) = B(x,x), x \in V$, from the set of symmetric bilinear forms on V into the set of quadratic forms on V is a $1 - 1$ correspondence. 5

(b) Let V be a complex vector space and H a form on V such that $H(\alpha,\alpha)$ is real for every $\alpha \in V$. Prove that H is hermitian. 5

(c) Let $A = \begin{bmatrix} -2 & 3 & 5 \\ 3 & 1 & -1 \\ 5 & -1 & 4 \end{bmatrix}$. Show that there exists an invertible matrix P such that P^tAP is diagonal. 4