**2023**
**M.Sc.**
**First Semester**
CORE – 01
**MATHEMATICS**
*Course Code: MMAC 1.11*
(Ordinary Differential Equations)

*Total Mark: 70* *Pass Mark: 28*
*Time: 3 hours*

*Answer five questions, taking one from each unit.*

## UNIT–I

1. (a) Consider the equation $y''+a_1y'+a_2y=0$, where $a_1, a_2$ are real. Suppose $\alpha+i\beta$ is a complex root of the characteristic polynomial, where $\alpha,\beta$ are real, $\beta\neq 0$. Then show that
   (i) $\alpha-i\beta$ is also a root
   (ii) any solution $\phi$ may be written in the form
   $\phi(x)=e^{\alpha x}(d_1\cos\beta x+d_2\sin\beta x)$ where $d_1, d_2$ are constants.
   Find the value of $\alpha,\beta$. 3+3+2=8

   (b) Let $I$ be the interval $0<x<1$. Find a function $\phi$ which has a continuous derivative on $-\infty<x<\infty$, which satisfies $y''=0$ in $I$ and $y''+k^2y=0$ outside $I$, $(k>0)$, and which has the form $\phi(x)=e^{ikx}+Ae^{-ikx}$, $(x\leq 0)$ and $\phi(x)=Be^{ikx}$, $(x\geq 1)$. Determine $\phi$ by computing the constants $A, B$ and its value in $I$. 6

2. (a) Let $\phi_n$ be any function satisfying the boundary value problem
   $y''+n^2y=0$, $y(0)=y(2\pi)$, $y'(0)=y'(2\pi)$, $(n=0,1,2,...)$ then show that $\int_0^{2\pi}\phi_n(x)\phi_m(x)=0$ if $n\neq m$. 4

   (b) Suppose $\phi_1,\phi_2$ are linearly independent solutions of the constant coefficient equation $y''+a_1y'+a_2y=0$, show that $W(\phi_1,\phi_2)$ is a constant if and only if $a_1=0$. 4

(c) Let $L(y) = y'' + a_1 y' + a_2 y$ where $a_1, a_2$ are constants, and let the characteristic polynomial be $p(r) = r^2 + a_1 r + a_2$. Now if $A$, $\alpha$ are constants and $p(\alpha) \neq 0$, show that there is a solution $\phi$ of $L(y) = Ae^{\alpha x}$ of the form $Be^{\alpha x}$ where $B$ is a constant and compute the particular solution of $L(y) = Ae^{\alpha x}$ if $p(\alpha) = 0$. 2+4=6

**UNIT–II**

3. (a) Prove that there exits $n$ linearly independent solution of $L(y) = 0$ on $I$. 4

(b) Consider the equation $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ for $x > 0$, then

(i) show that there is a solution of the form $x^r$, where $r$ is a constant

(ii) find the two solutions $\phi_1, \phi_2$ satisfying $\phi_1(1) = 1, \phi_2(1) = 0$; $\phi_1'(1) = 1, \phi_2'(1) = 1$. 4

(c) Consider the equation $y'' + \alpha(x) y = 0$, where $\alpha$ is a real valued continuous function for $0 < x < \infty$. Now, if $\alpha(x) \geq \varepsilon$ for $0 < x < \infty$, where $\varepsilon$ is a positive constant, show that every real valued solution has an infinity of zeros on $0 < x < \infty$. 6

4. (a) Let $\phi_1, \ldots, \phi_n$ be $n$ linearly independent solution of $L(y) = 0$ on an interval $I$. If $\phi$ is any solution of $L(y)$ on $I$ it can be represented in the form $\phi = c_1\phi_1 + \ldots + c_n\phi_n$, where $c_1, \ldots, c_n$ are $n$ constants. Then prove that any set of $n$ linearly independent solutions of $L(y) = 0$ on $I$ is a basis for the solution of $L(y) = 0$ on $I$. 4

(b) Consider the equation $y'' + \alpha(x) y = 0$ where $a$ is a continuous function on $-\infty < x < \infty$ which is of period $\xi > 0$ and let $\phi_1, \phi_2$ be the basis for the solutions satisfying $\phi_1(0) = 1, \phi_2(0) = 0$; $\phi_1'(0) = 1, \phi_2'(0) = 1$, then show that

(i) there is at least one non trivial solution $\phi$ of the period $\xi$ if and only if $\phi_1(\xi) + \phi_2'(\xi) = 2$. 3

(ii) there exist a one non trivial solution $\phi$ satisfying $\phi(x + \xi) = -\phi(x)$ if and only if $\phi_1(\xi) + \phi_2'(\xi) = -2$. 4

(c) Let $\phi, \psi$ be two real valued linearly independent solutions of $y''+\alpha(x)y=0$ on $a<x<b$, where $\alpha$ is a real valued. Show that between any two successive zeros of $\phi$ there is a zero of $\psi$. 3

## UNIT–III

5. (a) One solution of $x^2y''-xy'+y=0$, $(x>0)$ is $\phi_1(x)=x$. Find the solution $\psi$ of $x^2y''-xy'+y=x^2$, satisfying $\psi(1)=1$, $\psi'(1)=0$. 6

(b) Find two linearly independent power series solution of the equation $L(y)=y''-xy=0$. And show that the solutions are convergent. 8

6. (a) Find two linearly independent power series solution (in powers of $x$) of $y''-xy'+y=0$. 5

(b) The equation $y''+e^x y=0$ has a solution of the form

$\phi(x)=\sum_{k=0}^{\infty} c_k x^k$ which satisfies $\phi(0)=1$, $\phi'(0)=1$. Compute $c_0$, $c_1$, $c_2$, $c_3$, $c_4$, $c_5$. 4

(c) Show that $\int_{-1}^{1} P_n(x)P_m(x)dx=0, (n\neq m)$ where $P_n$ is the $n^{th}$ Legendre polynomial. 5

## UNIT–IV

7. (a) Solve the equation $y'=\frac{1}{2}\left(\frac{x+y-1}{x+2}\right)^2$ 3

(b) Find an integrating factor for each of the following equations and solve them

(i) $(2y^2+2)dx+3xy^2dy=0$ 3

(ii) $(5x^2y^2+2y)dx+(3x^4y+2x)dy=0$ 4

(c) Consider the equation $M(x,y)dx+N(x,y)dy=0$, where $M, N$ have continuous first partial derivatives on some rectangle $R$. Prove that the function $u$ on $R$, having continuous first partial derivatives, is an integrating factor if and only if $u\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=N\frac{\partial u}{\partial x}-M\frac{\partial u}{\partial y}$ on $R$. 4

8. (a) For each of the following problems compute the first four successive approximation: 3×3=9

(i) $y' = x^2 + y^2, y(0) = 0$ (ii) $y' = 1 + xy,\ y(0) = 1$

(iii) $y' = y^2,\ y(0) = 1$

(b) Consider the equation $y' = (3x^2+1)\cos^2 y + (x^2 - 2x)\sin 2y$ on the strip $s_a :| x | \leq a\ (a > 0)$. If $f(x, y)$ denotes the right side of this equation, show that $f(x, y)$ satisfies a Lipschitz condition on this strip, and hence show that every initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, has a solution which exit for all real $x$. 5

## UNIT–V

9. (a) Find all the solutions of the following equations for $| x | > 0$

(i) $x^2 y'' + xy' + 4y = 1$ 4×2=8

(ii) $x^2 y'' + xy' - 4\pi y = x$

(b) Let $L(y) = x^2 y'' + axy' + by$ where $a, b$ are constants, and let $q$ be the indicial polynomial $q(r) = r(r-1) + ar + b$. Then

(i) Show that the equation $L(y) = x^k$ has a solution $\psi$ of the form $\psi(x) = cx^k$ if $q(k) \neq 0$. 3

(ii) Suppose $k$ is a root of $q$ of multiplicity one. Show that there is a solution $\psi$ of $L(y) = x^k$ of the form $\psi(x) = cx^k \log x$. 3

10. (a) Find the solution near $x = 0$ of $x^2 y'' + (x + x^2) y' + (x - 9) y = 0$. 10

(b) If $J_n$ denotes the Bessel's function of first kind, then prove that $J_{-n}(x) = (-1)^n J_n(x)$. 4