2023
B.A./B.Sc.
Third Semester
GENERIC ELECTIVE – 3
MATHEMATICS
Course Code: MAG 3.11
(Vectors & Analytical Geometry)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If $\bar{r}(t)$ is a differentiable vector-valued function in 2-space or 3-space and $\|\bar{r}(t)\|$ is constant for all t, then prove that $\bar{r}(t).\bar{r}'(t)=0$. 3

   (b) Calculate $\frac{d}{dt}\left[\bar{r}_1(t).\bar{r}_2(t)\right]$ and $\frac{d}{dt}\left[\bar{r}_1(t)\times\bar{r}_2(t)\right]$ where $\bar{r}_1(t)=\cos t\hat{i}+\sin t\hat{j}+t\hat{k}$ and $\bar{r}_2(t)=\hat{i}+t\hat{k}$. 4

   (c) Find the arc length parametrization of the circular helix $\bar{r}=\cos t\hat{i}+\sin t\hat{j}+t\hat{k}$ that has reference point $\bar{r}(0)=(1,0,0)$ and the same orientation as the given helix. 3

   (d) Use the given information to find the position vector $\bar{r}(t)$ and velocity vector $\bar{v}(t)$ of the particle whose acceleration is given by $\bar{a}(t)=-\cos t\hat{i}-\sin t\hat{j};\bar{v}(0)=\hat{i};\bar{r}(0)=\hat{j}$. 4

2. (a) Find curvature $\kappa(t)$ and radius of curvature $\rho(t)$ for the circular helix $\bar{r}(t)=a\cos t\hat{i}+a\sin t\hat{j}+t\hat{k}$. 4

   (b) Find $\bar{T}(t)$, $\bar{N}(t)$ and $\bar{B}(t)$ at $t=0$ for the given vector $\bar{r}(t)=e^t\hat{i}+e^t\cos t\hat{j}+e^t\sin t\hat{k}$. 5

(c) If $\phi = 2x^3y^2z^4$, find $\nabla.(\nabla\phi)$. 3

(d) Determine the constant a so that the vector

$\bar{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal. 2

UNIT–II

3. (a) Using the given parametrization, evaluate the line integral

$\int_C (1+xy^2)\,ds$ 2×2=4

(i) $C : \bar{r}(t) = t\hat{i} + 2t\hat{j}\ \ (0 \le t \le 1)$

(ii) $C : \bar{r}(t) = (1-t)\hat{i} + (2-2t)\hat{j}\ \ (0 \le t \le 1)$

(b) Find the work done by the force field $\bar{F}(x, y) = xy^2\hat{k} + x^2y\hat{j}$ on a particle moving along an arbitrary smooth curve in the region from $P(1, 1)$ to $Q(0, 0)$. 4

(c) Verify Green's theorem for the integral $\oint_C (x^2 - y)dx + xdy$ where C is the circle $x^2 + y^2 = 4$. 4

(d) Evaluate the line integral $\int_C (3x+2y)\,dx + (2x-y)\,dy$ along the line segment from (0, 0) to (1, 1). 2

4. (a) Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F}(x, y) = \cos x\hat{i} + \sin x\hat{j}$ and

$C : \bar{r}(t) = t\hat{i} + t^2\hat{j}\ \ (-1 \le t \le 2)$. 2

(b) Determine whether $\bar{F}(x, y) = (\cos y + y\cos x)\hat{i} + (\sin x - x\sin y)\hat{j}$ is conservative. If so, find the potential function for it. 5

(c) Use Green's theorem to find the area enclosed by the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. 3

(d) Show that the given integral is independent of path and use fundamental theorem of line integral to find its value.

$\int_{(0,0)}^{\left(1, \frac{\pi}{2}\right)} \left(e^x \sin y\, dx + e^x \cos y\, dy\right)$ 4

UNIT–III

5. (a) Suppose that a curve lamina σ with constant density $\delta(x, y, z) = \delta_0$ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$. Find the mass of the lamina. 4

(b) Evaluate the surface integral $\iint_\sigma x^2 dx$, where σ is the part of the surface of sphere $x^2 + y^2 + z^2 = 1$. 5

(c) Verify Stoke's theorem for the vector field

$\bar{F}(x, y, z) = z^2\hat{i} + 2x\hat{j} - y^3\hat{k}$, taking C to be the circle $x^2 + y^2 = 1$ with the counterclockwise orientation looking down the positive z-axis. 5

6. (a) Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane and suppose that σ is oriented up. Find the flux of the vector field $\bar{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ across σ. 4

(b) Use Stoke's theorem to find the work performed by the force field

$\bar{F}(x, y, x) = x^2\hat{i} + 4xy^3\hat{j} + y^2x\hat{k}$ on a particle that traverses the rectangle C in the plane $z = y$. 5

(c) Use divergence theorem to find the outward flux of the vector field

$\bar{F}(x, y, z) = x^3\hat{i} + y^3\hat{j} + z^2\hat{k}$ across the surface of the region enclosed by the circular cylinder $x^2 + y^2 = 9$ and the plane $z = 0$ and $z = 2$. 5

UNIT–IV

7. (a) If $ax + by$ is transform to $a'x' + b'y'$ due to rotation of axes, then show that $a^2 + b^2 = a'^2 + b'^2$. 4

(b) Reduce the following equation to its canonical form

$x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$. 5

(c) Prove that the equation of the tangent at any point $P(h, k)$ on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$. 5

8. (a) Transform the equation $3x^2 + 2xy + 3y^2 - 1 = 0$ if the axes are rotated through an angle 45°. 3

(b) Prove that the length of the focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is inclined at an angle θ to the major axis is

$$\frac{2ab^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}.$$ 7

(c) Prove that the tangent at the extremities of a pair of conjugate diameters of an ellipse forms a parallelogram with constant area. 4

UNIT–V

9. (a) Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. 4

(b) Find the equation of the plane passing through the points (1, 2, 3), (2, 3, 4) and perpendicular to the plane $2x + 3y - 4z = 9$. 5

(c) Show that the straight line through the points (a, b, c) and (α, β, γ) passes through the origin if $\alpha a + \beta b + \gamma c = Pp$, where P and p are the distances of the points from the origin. 5

10. (a) If A, B, C are three points on the coordinate axes such that $OA = a$, $OB = b$ and $OC = c$ where O is the origin. Find the point which is equidistant from A, B, C and origin. 5

(b) Find the equation of the image line of the line $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$ in the plane $2x - y + z + 6 = 0$. 5

(c) Prove that the angle between the lines whose direction cosines satisfy $l + m + n = 0$ and $l^2 = m^2 + n^2$ is $\frac{\pi}{3}$. 4