**2023**

**B.A./B.Sc.**

**Fifth Semester**

CORE – 12

# MATHEMATICS

*Course Code: MAC 5.21*

(Group Theory - II)

*Total Mark: 70* *Pass Mark: 28*

*Time: 3 hours*

*Answer five questions, taking one from each unit.*

## UNIT–I

1. (a) Define inner automorphism of a group $G$ and determine the set of all inner automorphisms of the dihedral group $D_4$ of order 8. 4

(b) Let $H$ be a normal subgroup of $K$ and $K$ be a characteristic subgroup of $G$. Then give an example to show that $H$ need not be normal in $G$. 4

(c) Let $a,b,c \in G$. Prove that $[a,bc]=[a,c]\left(c^{-1}[a,b]c\right)$. 3

(d) If $p$ is an odd prime, then prove that the automorphism group of the cyclic group of order $p$ is of order $p-1$. 3

2. (a) If a group $G$ is isomorphic to $H$, prove that Aut($G$) is isomorphic to Aut($H$). 4

(b) Let $G$ be a group with subgroups $H$ and $K$ with $H \leq K$. 3+3=6

(i) Prove that if $H$ is characteristic in $K$ and $K$ is normal in $G$ then $H$ is normal in $G$.

(ii) Prove that if $H$ is characteristic in $K$ and $K$ is characteristic in $G$ then $H$ is characteristic in $G$.

(c) Prove that if $x, y \in G$ then $[y, x] = [x, y]^{-1}$. Deduce that for any subsets $A$ and $B$ of $G$, $[A, B] = [B, A]$. 4

## UNIT–II

3. (a) Let $s$ and $t$ be relatively prime. Prove that $U(st)$ is isomorphic to the external direct product of $U(s)$ and $U(t)$. 5

(b) Prove or disprove: 4

Any finite group $G$ of prime-power order can be written in the form $\langle a \rangle \times K$, where $a$ is an element of maximum order in $G$ and $K$ is a subgroup of $G$.

(c) Show that the multiplicative group of nonzero real numbers is an internal direct product of two non trivial subgroups. 5

4. (a) Prove that the internal direct product of two subgroups $H$ and $K$ of a group $G$ is isomorphic to the external direct product of $H$ and $K$. 5

(b) Prove that any abelian group of order 45 has an element of order 15. 4

(c) Determine the number of automorphisms of order 12 in $\mathrm{Aut}(\mathbb{Z}_{720})$. 5

## UNIT–III

5. (a) Let $G$ be the dihedral group $D_4$ of order 8. 3+4=7

(i) If $A$ is the set of all left cosets of a subgroup $H$ in $G$, show that $G$ acts on $A$.

(ii) Let $H$ and $V$ be the horizontal and vertical axes of symmetries of a square. Show that $G$ acts on $\{H, V\}$ and determine the kernel of this action.

(b) Prove that the symmetric group $S_5$ acts transitively in its usual action as permutation on $A = \{1, 2, 3, 4, 5\}$. 3

(c) Show that the kernel of an action of the group $G$ on the set $A$ is the same as the kernel of the corresponding permutation representation $G \to S_A$. 4

6. (a) Define kernel of the action of a group $G$ on a set $A$. Show that the additive group $\mathbb{R}$ acts on the $xy$-plane $\mathbb{R} \times \mathbb{R}$ by $r.(x, y) = (x + ry, y)$, where $r \in \mathbb{R}$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$. 1+3=4

(b) Let $G$ be a finite group acting on a set $A$. For each $a \in A$, let $\mathcal{O}_a$ denote the orbit of $G$ containing $a$ and $G_a$ denote the stabilizer of $a$ in $G$. Prove that $|G| = |G_a||\mathcal{O}_a|$ 5

(c) Let $G = S_n$, fix $i \in \{1, 2, 3, \ldots, n\}$ and let $G_i = \{\sigma \in G \mid \sigma(i) = i\}$. Use group action to prove that $G_i$ is a subgroup of $G$. Find $|G_i|$. 5

## UNIT–IV

7. (a) If $G$ is a group of odd order, prove for any non-identity element $x \in G$ that $x$ and $x^{-1}$ are not conjugate in $G$. 5

   (b) For a group $G$ with $|G| > 1$, verify if $G$ acts transitively on itself by conjugation. Also, prove that the one element set $\{a\}$ is a conjugacy class if and only if $a$ is in the centre of $G$. 2+3=5

   (c) Let $\sigma$ be an $m$-cycle in the symmetric group $S_n$. Then if $C_{S_n}(\sigma)$ is the centralizer of $\sigma$, prove that $C_{S_n}(\sigma) = m.(n-m)!$ 4

8. (a) Determine the class equation for non-abelian groups of order 39 and 55. 5

   (b) Find all finite groups up to isomorphism which have exactly two conjugacy classes. 4

   (c) Let $\sigma \in A_n$. Show that all elements in the conjugacy class of $\sigma$ in $S_n$ are conjugate in $A_n$ if and only if $\sigma$ commutes with an odd permutation. 5

## UNIT–V

9. (a) Show that the centre of a group of order 60 cannot have order 4. 4

   (b) Let $n$ be an odd number greater than 1. Prove that there does not exist a simple group of order $2.n$. 5

   (c) Define Sylow $p$-subgroup of a group $G$. Let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is normal in $G$, prove that all subgroups generated by elements of $p$-power order are $p$-groups. 1+4=5

10. (a) How many Sylow 5-subgroups does $S_5$ have? Exhibit two. 4

   (b) Let $G$ be a group of order 60. If the Sylow 3-subgroup is normal, show that the Sylow 5-subgroup is normal. 5

   (c) Find all normal subgroups of $S_n$ for all $n \geq 5$. 5