2023
B.A./B.Sc.
Third Semester
CORE – 7
MATHEMATICS
Course Code: MAC 3.31
(PDE & Systems of ODE)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Find the PDE arising from the surface $z = xy + f(x^2 + y^2)$. 3
   (b) Find the general solution of the linear PDE $x^2u_x + y^2u_y = (x + y)u$. 5
   (c) Find the solution of the Cauchy problem $xu_x + yu_y = 2xy$ with $u = 2$ on $y = x^2$. 6

2. (a) Show that two parameter family of curves $u - ax - by - ab = 0$ satisfies the non linear equation $xp + yq + pq = u$. 3
   (b) Use the method of separation of variables $u(x, y) = f(x)\, g(x)$ to solve $u_x = 2u_y + u$, $u(x, 0) = 6e^{-3x}$. 5
   (c) Reduce the equation $yu_x + u_y = x$ to canonical form and obtain the general solution. 3+3=6

UNIT–II

3. (a) Determine the region in the xy-plane where the second order PDE $u_{xx} + yu_{yy} + \frac{1}{2} u_y = 0$ is hyperbolic. 3
   (b) Derive one dimensional wave equation $u_u - c^2u_{xx} = 0$. 6
   (c) Find the general solution of $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$. 5

4. (a) Find the region in which the equation $yu_{xx} + 3yu_{xy} + 3u_x = 0, y \neq 0$ is hyperbolic. Find the general solution by reducing it to canonical form. 1+5+2=8
   (b) Derive the one dimensional heat equation. 6

UNIT–III

5. (a) Obtain the D'Alembert's solution of the Cauchy problem of the one dimensional wave equation defined as follows: 8

$$u_{u} - c^2 u_{xx} = 0, x \in \mathbb{R}, t > 0$$
$$u(x,0) = f(x), x \in \mathbb{R}$$
$$u_t(x,0) = g(x), x \in \mathbb{R}$$

(b) Determine the solution of the initial value problem: 6

$u_{tt} - c^2 u_{xx} = e^x$, $u(x, 0) = 5$, $u_t(x,0) = x^2$

6. (a) Solve the heat conduction equation

$$u_t = k u_{xx}, 0 < x < l, t > 0$$
$$u(0,t) = 0, u(l,t) = 0, t \geq 0$$
$$u(x,0) = f(x), 0 \leq x \leq l$$

by the method of separation of variables. 8

(b) Determine the solution of the initial boundary value problem: 6

$$u_{tt} = c^2 u_{xx}, 0 < x < l, t > 0$$
$$u(x,0) = \sin\left(\frac{\pi x}{l}\right), 0 \leq x \leq l$$
$$u_t(x,0) = 0, 0 \leq x \leq l$$
$$u(0,t) = 0, u(l,t) = 0, t \geq 0$$

UNIT–IV

7. (a) Solve the following system of equations by an operator method: 6

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$
$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$$

(b) Two tanks *X* and *Y* are interconnected. Tank *X* initially contains 100 litres of brine in which there is dissolved 5 kg of salt, and tank *Y* initially contains 100 litres of brine in which there is dissolved 2 kg of salt. Starting at time $t = 0$, 8

(i) pure water flows into tank *X* at the rate of 6 litres/min,

(ii) brine flows from tank X into tank Y at the rate of 8 litres/min,

(iii) brine is pumped from tank Y back into tank X at the rate of 2 litres/min

(iv) brine flows out of tank Y and away from the system at the rate of 6 litres/min. The mixture in each tank is kept uniform by stirring. How much salt is in each tank at any time $t > 0$?

8. (a) Consider the homogeneous linear system $\left.\begin{aligned} \frac{dx}{dt} &= 5x+3y \\ \frac{dy}{dt} &= 4x+y \end{aligned}\right\}$.

(i) Show that $\left.\begin{aligned} x &= 3e^{7t} \\ y &= 2e^{7t} \end{aligned}\right\}$ and $\left.\begin{aligned} x &= e^{-t} \\ y &= -2e^{-t} \end{aligned}\right\}$ are linearly independent solutions $\forall t \in [a,b]$. 2

(ii) Write the general solution of the homogeneous linear system. Also find the solutions $x = f(t), y = g(t)$ such that $f(t) = 0$, $g(0) = 8$. 1+3=4

(b) Find the general solution of the $\left.\begin{aligned} \frac{dx}{dt} &= 3x+2y \\ \frac{dy}{dt} &= -5x+y \end{aligned}\right\}$. 8

UNIT–V

9. (a) Use the method of successive approximations to find first three members $\varphi_1, \varphi_2, \varphi_3$ of a sequence of functions that approaches the solution of the initial-value problem: $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$. 8

(b) By using an appropriate example, show that the improved Euler method is a better approximation than the Euler's method. 6

10. (a) Discuss Euler method to approximate the solution of the initial -value problem: $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. 8

(b) Consider the initial value problem: $\frac{dy}{dx} = x + y, y(0) = 1$. Apply the fourth order Runge-Kutta method to approximate the value (up to the 5[th] decimal places) of the solution y at $x = 0.1$ when $h = 0.1$. 6
