# 2023
# B.A./B.Sc.
## Third Semester
CORE – 6
## MATHEMATICS
*Course Code: MAD 3.21*
(Group Theory - I)

*Total Mark: 70* *Pass Mark: 28*
*Time: 3 hours*

*Answer five questions, taking one from each unit.*

## UNIT–I

1. (a) Define abelian group and prove that the set of all positive rational numbers forms an abelian group under the operation defined by $a * b = \frac{ab}{2}$. 5
   (b) Define inverse of an element of a group $G$ and prove that the inverse of each element of a group is unique. 5
   (c) Let $a$ and $b$ be elements of an abelian group and let $n$ be any integer. Show that $(ab)^n = a^n b^n$. 4

2. (a) Prove that the set of all 2×2 matrices with real entries under matrix addition forms an abelian group. 5
   (b) Prove that a group in which every element is its own inverse is abelian. 4
   (c) Define identity element of a group $G$ and prove that the identity element of a group is unique. 5

## UNIT–II

3. (a) Define subgroup of a group and prove that a non empty subset $H$ of a group $G$ is a subgroup of $G$ if and only if $a, b \in H \Rightarrow ab^{-1} \in H$. 5
   (b) Define centre of a group and show that it is a subgroup of the group. 5

(c) Suppose that $\langle a \rangle, \langle b \rangle$ and $\langle c \rangle$ are cyclic groups of order 6, 8 and 20 respectively. Find all generators of $\langle a \rangle, \langle b \rangle$ and $\langle c \rangle$. 4

4. (a) Show that the union of two subgroups of a group is a subgroup of the group if and only if one is contained in the other. 6

(b) Define cyclic group and prove that an infinite cyclic group has exactly two generators. 5

(c) Find the centre of the group of all 2×2 non singular matrices over real numbers. 3

## UNIT–III

5. (a) Prove that the set of all permutations on a finite set $S$ having $n$ elements, is a group under composition of functions. 5

(b) Prove that any two right cosets of a subgroup $H$ in a group $G$ are either identical or disjoint. 5

(c) Show by an example that the converse of Lagrange's theorem may not hold. 4

6. (a) Define right coset and prove that if $H$ is a subgroup of a group $G$, then $Ha = H \Leftrightarrow a \in H$. 5

(b) State and prove Lagrange's theorem for finite groups. 5

(c) If $H$ is a subgroup of the permutation group $S_n \ (n \geq 2)$, show that either all permutations in $H$ are even or exactly half are even. 4

## UNIT–IV

7. (a) Define external direct product of two groups and prove that if $G_1$ and $G_2$ are two groups, then $G_1 \times G_2$ is also a group under the operation defined by $(g_1, g_2).(h_1, h_2) = (g_1 h_1 g_2 h_2)$. 5

(b) Prove that a subgroup $N$ of a group $G$ is a normal subgroup of $G$ if and only if every left coset of $N$ in $G$ is also a right coset of $N$ in $G$. 5

(c) If a cyclic subgroup $H$ of a group $G$ is normal in $G$, prove that every subgroup of $H$ is normal in $G$. 4

8. (a) If $G_1$ and $G_2$ are two groups, show that $G_1 \times G_2$ is abelian if and only if both $G_1$ and $G_2$ are abelian. 5

(b) Let $H$ be a normal subgroup of a group $G$ and $K$ be any subgroup of $G$. Then show that $HK = \{hk(h \in H, k \in K\}$ is a subgroup of $G$. 4

(c) Show that every quotient group of a cyclic group is cyclic. Is the converse true? 5

## UNIT–V

9. (a) Define group homomorphism and show that for any two groups $G$ and $G'$, if $f : G \to G'$ is a group homomorphism, then $f(G) = \{f(a) : a \in G\}$ is a subgroup of $G$. 3

(b) State and prove Cayley's theorem. 7

(c) Determine all homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$. 4

10. (a) State and prove the fundamental theorem of group homomorphism. 6

(b) If $f : G \to G'$ is a group homomorphism, prove that $f$ is one-one if and only if Ker $f = \{e\}$, where Ker $f$ denotes the kernel of $f$. 5

(c) Show that an infinite cyclic group is isomorphic to the group of integers. 3