2023
B.A./B.Sc.
Third Semester
CORE – 5
MATHEMATICS
Course Code: MAC 3.11
(Theory of Real Functions)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has a cluster point. Justify. 4

(b) Use the definition of limit to show that $\lim_{x\to 6} \frac{x^2 - 3x}{x+3} = 2$. 5

(c) Let f be defined on $(0,\infty)$ to $\mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$. 5

2. (a) Show that $\lim_{x\to 0}\left(x + \operatorname{sgn}(x)\right)$ does not exist. 5

(b) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose that $f(x) \geq 0$ for all $x \in A$ and let $\sqrt{f}$ be the function defined for $x \in A$ by $\sqrt{f}(x) := \sqrt{f(x)}$. If $\lim_{x\to c} f$ exists, prove that $\lim_{x\to c}\sqrt{f} = \sqrt{\lim_{x\to c} f}$. 5

(c) Let f and g be defined on $(0,\infty)$ and suppose that $\lim_{x\to\infty} f = L$ and $\lim_{x\to\infty} g = \infty$. Prove that $\lim_{x\to\infty} f \circ g = L$. 4

UNIT–II

3. (a) Establish the discontinuity criterion. 5

(b) A function $f:\mathbb{R}\to\mathbb{R}$ is said to be additive if $f(x+y)=f(x)+f(y), \forall x,y\in\mathbb{R}$. Prove that if *f* is continuous at some point x_0 then it is continuous at every point of $\mathbb{R}$. 5

(c) Show that the function $f(x):=\dfrac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$. 4

4. (a) If $[a,b]$ is a closed and bounded interval and $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, then show that *f* has an absolute maximum on $[a,b]$. 5

(b) Use the nonuniform continuity criterion, to show that $\sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0,\infty)$. 5

(c) Let $I:=[a,b]$ and let $f:I\to\mathbb{R}$ be a continuous function on I such that for each *x* in *I* there exists *y* in *I* such that $|f(y)|\le\frac{1}{2}|f(x)|$. Prove that there exists a point $c\in I$ such that $f(c)=0$. 4

UNIT–III

5. (a) Find the points of relative extrema of the following functions:

(i) $f(x):=\dfrac{x}{x^2+1}$ for $x\in\mathbb{R}$ 2½×2=5

(ii) $g(x):=\left|x^2-1\right|$

(b) Establish the chain rule. 7

(c) Differentiate and simplify the function $f(x):=\left(\sin x^k\right)^m$ for $m,k\in\mathbb{N}$. 2

6. (a) Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Show that if $\lim_{x\to a} f'(x)=A$, then $f'(a)$ exists and equals A. 5

(b) Let $I\subseteq\mathbb{R}$ be an interval, $c\in I$, $f,g:I\to\mathbb{R}$ be functions that are differentiable at c. Then show that function fg is differentiable at c and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$. 5

(c) Let I be an interval and let $f:I\to\mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I. 4

UNIT–IV

7. (a) State and prove the mean value theorem. 5

(b) Use the mean value theorem to obtain an approximate estimate of $\sqrt{111}$. 5

(c) Evaluate : $\lim_{x\to\infty} x^{\frac{1}{x}}$ in $(0,\infty)$. 4

8. (a) Show $ex\le e^x, \forall x\in\mathbb{R}$. 5

(b) Let f and g be differentiable on an open interval I and consider $a\in I$. Define h on I by the rules: $h(x)=f(x)$ for $x<a$ and $h(x)=g(x)$ for $x\ge a$. Prove h is differentiable at a if and only if both $f(a)=g(a)$ and $f'(a)=g'(a)$ hold. 5

(c) Let f be defined on $\mathbb{R}$, and suppose $|f(x)-f(y)|\le(x-y)^2$ for all $x,y\in\mathbb{R}$. Prove f is a constant function. 4

UNIT–V

9. (a) Approximate the number e with error less than 10^{-5}. 5

(b) Use Taylor's theorem to show that $1-\frac{1}{2}x^2\le\cos x$ for all $x\in\mathbb{R}$. 5

(c) Let $f(x) := \cos ax$ for $x \in \mathbb{R}$ where $a \neq 0$. Find $f^{(n)}$ for $n \in \mathbb{N}, x \in \mathbb{R}$. 4

10. (a) Let $I \subseteq \mathbb{R}$ be an open interval, let $f: I \to \mathbb{R}$ be differentiable on I, and suppose that $f''(a)$ exists at $a \in I$. Show that $f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$. Give an example where this limit exists, but the function does not have a second derivative at a. 5

(b) Compute Taylor's series for $\cos x$ about 0. 5

(c) Obtain the inequality $e^{\pi} > \pi^{e}$ applying Taylor's theorem. 4