2023

B.A./B.Sc.

First Semester

CORE – 2

MATHEMATICS

Course Code: MAC 1.21

(Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Find all the complex numbers z such that $|z|=1$ and $\left|\frac{z}{\overline{z}}+\frac{\overline{z}}{z}\right|=1$. 5

   (b) Prove that $\sin 5t = 16\sin^5 t - 20\sin^3 t + 5\sin t$ and
   $\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t$ 5

   (c) Find the third root of the number $z = 1 - i$. 4

2. (a) Suppose $z_1 = r_1(\cos t_1 + i\sin t_1)$ and $z_2 = r_2(\cos t_2 + i\sin t_2) \neq 0$,
   then show that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(t_1 - t_2) + i\sin(t_1 - t_2)]$. 5

   (b) Show that the common roots of $z^m - 1 = 0$ and $z^n - 1 = 0$ are the roots of $z^d - 1 = 0$, where $d = \gcd(m, n)$. 5

   (c) Compute $(1 + i)^{1000}$. 4

UNIT–II

3. (a) For a fixed $k \in \mathbb{N}$, define a relation R on $\mathbb{Z}$ as $'mRn \Leftrightarrow m-n$ is an integral multiple of k. Prove that R is an equivalence relation. 5

   (b) Show that $(0, \infty)$ and $\mathbb{R}$ have the same cardinality. 5

   (c) Find the remainder when 41^{65} is divided by 7. 4

4. (a) Define $f : \mathbb{N} \to \mathbb{N}$ by $f(m) = \begin{cases} m-1 & \text{if } m \text{ is even} \\ m+1 & \text{if } m \text{ is odd} \end{cases}$ show that f is a bijection and find its inverse. 5

(b) Using the principle of mathematical induction, prove that 5

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

(c) Using division algorithm establish that the square of any integer is either of the form $3k$ or $3k+1$. 4

UNIT–III

5. (a) Determine if $\mathbf{b}=\begin{bmatrix}3\\-7\\-3\end{bmatrix}$ is a linear combination of the vectors formed by the columns of the matrix $A=\begin{bmatrix}1 & -4 & 2\\0 & 3 & 5\\-2 & 8 & -4\end{bmatrix}$. 5

(b) Determine if the columns of the following matrix span $\mathbb{R}^4$: 5

$$\begin{bmatrix}12 & -7 & 11 & -9 & 5\\-9 & 4 & -8 & 7 & -3\\-6 & 11 & -7 & 3 & -9\\4 & -6 & 10 & -5 & 12\end{bmatrix}$$

(c) Determine if the following system is consistent: 4

$$x_2-4x_3=8$$
$$2x_1-3x_2+2x_3=1$$
$$4x_1-8x_2+12x_3=1$$

6. (a) Apply elementary row operations to transform the following matrix first into its echelon form and then into reduced echelon form: 5

$$\begin{bmatrix}0 & 3 & -6 & 6 & 4 & -5\\3 & -7 & 8 & -5 & 8 & 9\\3 & -9 & 12 & -9 & 6 & 15\end{bmatrix}$$

(b) If A is an $m\times n$ matrix, then show that, for each $b\in\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution if and only if A has a pivot position in every row. 5

(c) Suppose an economy has only two sectors, goods and services. Each year, goods sells 80% of its output to services and keeps the rest, while services sells 70% of its output to goods and retains the rest. Find equilibrium prices for the annual outputs of the goods and services sectors that make each sector's income match its expenditures. 4

UNIT–IV

7. (a) Show that an indexed set $S = \{\mathbf{v}_1,..., \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. 5

(b) Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? 3+2=5

(c) Find the inverse of the matrix: 4

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

8. (a) Define a linear transformation. If T is a linear transformation, then prove that: 1+2+2=5

(i) $T(0) = 0$

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$, for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

(b) Let A be a square matrix, prove that A is an invertible matrix if and only if A^T is an invertible matrix. 5

(c) Find the value(s) of h, justifying your answer, for which the following vectors are linearly dependent: 4

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$$

UNIT–V

9. (a) Show that similar matrices have the same charateristic polynomial. Is the converse true? Justify. 5

(b) Show that 2 is an eigenvalue of the matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis for the corresponding eigenspace. 2+3=5

(c) Determine the rank of the matrix: 4

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

10. (a) If $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$, then find a basis for *Col A* and a basis for *Nul A*. 5

(b) Prove that an $n \times n$ matrix A is invertible if and only if 0 is not an eigenvalue of A. 5

(c) Show that the null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. 4
