2023

B.A./B.Sc.

First Semester

CORE – 1

MATHEMATICS

Course Code: MAC 1.11

(Calculus)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Find the first derivative of the following with respect to x 2+2=4

(i) $\tanh^{-1}\left(\dfrac{2}{2+x}\right)$

(ii) $\log\sinh\left(x+\cosh^2 x\right)$

(b) If $y^{\frac{1}{m}}+y^{\frac{-1}{m}}=2x$, then prove that

$\left(x^2-1\right)y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2-m^2\right)y_n=0$. 5

(c) Find the horizontal and vertical asymptotes of the function

$f(x)=\dfrac{x^3-2}{|x^3|+1}$. 3

(d) If $x=\log\phi$ and $y=\phi^2-1$, find $\dfrac{d^2y}{dx^2}$. 2

2. (a) Consider the function $f(x) = x^4 - 5x^3 + 9x^2$. Find 1×5=5

(i) The intervals on which f is increasing,

(ii) The intervals on which f is decreasing,

(iii) The intervals on which f is concave up,

(iv) The intervals on which f is concave down, and

(v) The x-coordinates of all inflection points.

(b) Find the n^{th} derivatives of $y = \frac{x^3}{x^2 - 1}$. Also, prove that at $x = 0$ 5

$$y_n = \begin{cases} 0 & \text{, when } n \text{ is even} \\ -(n!) & \text{, when } n \text{ is odd and greater than 1} \end{cases}$$

(c) State Leibnitz's theorem and use it to find the third derivatives of the function $y = e^{2x} \log x$. 1+3=4

UNIT–II

3. (a) Give a graph of the polynomial $f(x) = 2x^3 - 3x^2 - 36x + 5$ and label the coordinates of the intercepts, stationary points, and inflection points. 5

(b) A rectangular plot of farm land will be bounded on one side of a river and on the other three sides by a single-stranded electric fence. With 600 meters of wire at your disposal, what is the largest area you can enclose, and what are its dimensions? 5

(c) Find $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x}}$. 4

4. (a) Sketch the curve in polar coordinates $r^2 = 4\cos 2\theta$. 5

(b) It is projected that t years from now, the population of a certain country will be $P(t) = 50e^{0.02t}$ millions. 3+2=5

(i) At what rate will the population be changing with respect to time 10 years from now?

(ii) At what percentage rate will the population be changing with respect to time t years from now?

(c) Trace the graph of

Cycloid: $x = 2(\theta + \sin\theta), y = 2(1 - \cos\theta), -\pi \le \theta \le \pi$ and indicate the direction of θ increasing. 4

UNIT–III

5. (a) Derive the reduction formula for $\int \cos^n x dx$ and evaluate

$\int_0^{\pi/2} \cos^n x dx$. 5

(b) Use washer method to find the volume of the solid that results when the region enclosed around the curve $y=\sqrt{25-x^2}$ and $y = 3$ is revolved around the x-axis. 5

(c) Show that the total arc length of the ellipse $x = a\cos t, y = b\sin t$, $0 \le t \le 2\pi$ for $a > b > 0$ is given by $4a\int_0^{\pi/2}\sqrt{1-k^2\cos^2 t}\,dt$, where $k=\dfrac{\sqrt{a^2-b^2}}{a}$. 4

6. (a) Find the area of the surface generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis. 5

(b) Derive the reduction formula for $\int x^m(\log x)^n dx$ and evaluate $\int_0^1 x^m(\log x)^n\,dx$, when $m, n \ge 0$. 5

(c) The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume. 4

UNIT–IV

7. (a) Identify and sketch the graph $xy = 1$. 5

(b) Sketch the graph of $r=\dfrac{2}{1-\cos\theta}$ in polar coordinates. 5

(c) Find the equation to the curve $9x^2 + 4y^2 + 18x - 16y - 11 = 0$ referred to the parallel axes through the point $(-1, 2)$. 4

8. (a) Find the angle through which the axes are to be rotated so that the equation $17x^2 - 18xy - 7y^2 = 1$ may be reduced to the form $Ax^2 + By^2 = 1$, $A > 0$. Also, find A and B. 5

(b) Reduce the following equation to its canonical form 5
$7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$.

(c) Find an equation in xy-coordinate for the ellipse with center $(-3, 2)$, vertex $(2, 2)$ and eccentricity $e=\dfrac{4}{5}$. 4

UNIT–V

9. (a) Prove or disprove: 1½+1½+2=5

Suppose that $\vec{a} \neq \vec{0}$.

(i) If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, then $\vec{a} = \vec{c}$

(ii) If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then $\vec{b} = \vec{c}$

(iii) If $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then $\vec{b} = \vec{c}$

(b) Suppose that a particle moves through 3-space so that its position vector at time t is $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$. Find the scalar tangential and normal components of acceleration at time $t = 1$. 5

(c) Evaluate the integral 2+2=4

(i) $\int_{-3}^{3}\left\langle (3-t)^{3/2}, (3+t)^{3/2}, 1\right\rangle dt$

(ii) $\int\left(\cosh t\,\hat{i} + \sinh t\,\hat{j} + \frac{1}{t}\hat{k}\right)dt$

10. (a) Let $\vec{u} = \vec{u}(t), \vec{v} = \vec{v}(t)$ and $\vec{w} = \vec{w}(t)$ be differentiable vector-valued functions then show that 4

$$\frac{d}{dt}\left[\vec{u}.(\vec{v}\times\vec{w})\right] = \frac{d\vec{u}}{dt}.[\vec{v}\times\vec{w}] + \vec{u}.\left[\frac{d\vec{v}}{dt}\times\vec{w}\right] + \vec{u}.\left[\vec{v}\times\frac{d\vec{w}}{dt}\right].$$

(b) A shell is fired from ground level at an elevation of angle α and a muzzle speed v_0. Show that the maximum height reached by the shell is $\frac{(v_0 \sin\alpha)^2}{2g}$. 5

(c) A rock is thrown downward from the top of a building, 168 ft high, at an angle 60° with the horizontal. How far from the base of the building will the rock land if its initial speed is 80 ft/s? 5