2022

M.Sc.

Third Semester

DISCIPLINE SPECIFIC ELECTIVE – 02

MATHEMATICS

Course Code: MMAD 3.21

(Tensor Analysis & Riemannian Geometry)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

(*Symbols have their usual meaning*.)

UNIT–I

1. (a) If A_r^{pq} and B_r^{pq} are tensors, then prove that their sum and difference are tensors. Also prove that the tensor product of the tensors of type (r, s) and (r', s') is a tensor of type $(r + r', s + s')$. 3+3=6

   (b) Define inner product and show that the inner product of the tensors A_r^p and B_t^{qs} is a tensor of rank three. 4

   (c) If A^{ij} and A_{ij} are components of symmetric relative tensors of weight w, then show that 2×2=4

   (i) $\left|\overline{A}^{ij}\right| = \left|A^{ij}\right| \cdot \left|\dfrac{\partial x}{\partial \overline{x}}\right|^{w-2}$

   (ii) $\left|\overline{A}_{ij}\right| = \left|A_{ij}\right| \cdot \left|\dfrac{\partial x}{\partial \overline{x}}\right|^{w+2}$

2. (a) Define relative tensor, relative vector and relative scalar. Prove that the equation of transformation of a relative tensor possess the group property. 1+1+1+2=5

   (b) Prove that the set of n^3 functions A^{ijk} form the components of a tensor if $A^{ijk} B_{ij}^p = C^{pk}$ provided that B_{ij}^p is an arbitrary tensor and C^{pk} a tensor. What happens if B_{ij}^p is symmetrical in i and j? 4+1=5

(c) If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, Show that $C_{ij}+C_{ji}$ is a covariant tensor of the second order. 4

UNIT–II

3. (a) Show that
   (i) $g_{ij}dx^i dx^j$ is an invariant. 1
   (ii) the magnitude of any vector $\overline{u}$ is zero if the projections of a $\overline{u}$ on e_{h1} are all zero. 3
   (b) If θ is the angle between two vectors $\overline{u}$ and $\overline{v}$ at a point O of a Riemannian V_n, then find the value of $\cos\theta$. Also, find the angle between two coordinate curves. 3+2=5
   (c) Find the metric of a Euclidean space referred to the cylindrical coordinates. 5

4. (a) Prove that an arbitrary V_n does not admit an *n*-ply orthogonal system of hypersurfaces. 4
   (b) Define Euclidean space and show that the condition that a V_n be immersible in a S_m is that $m \geq \frac{n}{2}(n+1)$. 1+4=5
   (c) Calculate the quantities g^{ij} for a V_3 whose fundamental form in coordinates *u*, *v*, *w* is

   $$a\,du^2 + b\,dv^2 + c\,dw^2 + 2f\,dv\,dw + 2g\,dw\,du + 2h\,du\,dv$$ 5

UNIT–III

5. (a) Prove that
   (i) the Christoffel symbols $[jk,i]$ and $\left\{\begin{matrix} i \\ jk \end{matrix}\right\}$ are symmetric in j and *k*. 2
   (ii) $[ij,k]+[jk,i]=\partial_j g_{ki}$ and $\frac{\partial gik}{\partial x^j} = -g^{hk}\left\{\begin{matrix} i \\ hj \end{matrix}\right\} - g^{hi}\left\{\begin{matrix} k \\ hj \end{matrix}\right\}$ 3

(b) Obtain covariant derivative of a covariant vector w.r.t. the fundamental tensor g_{ij}. 4

(c) Find the Christoffel symbols of first and second kind for the V_2 with line element $ds^2 = a^2 \left(dx^1\right)^2 + a^2 \sin^2 x^1 \left(dx^2\right)^2$, a is a constant. 5

6. (a) Show that the covariant derivative of a second rank mixed tensor is a tensor of rank three. 8

(b) If A_{ij} is the curl of a covariant vector, prove that

$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$ and that this is equivalent to

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0$$ 6

UNIT–IV

7. (a) Define curvature of a curve and principal normal. 3

(b) Prove the necessary and sufficient condition that a system of coordinates be geodesic with pole at P_0 are that their second covariant derivatives with respect to the metric of the space all vanish to P_0. 4

(c) Show that the geodesics are straight lines in S_n, the Euclidean space of n dimensions, and also determine the distance formula in S_n. 7

8. (a) Find the necessary and sufficient condition for a vector v^i of variable magnitude to suffer a parallel displacement along the curve C. 10

(b) Prove that the angle between any two vectors is the same whether it is calculated w.r.t. V_n or V_m. 4

UNIT–V

9. (a) Obtain the expression for Riemannian symbol of first kind. 4

(b) State and prove the symmetric property and differential property of R_{hijk}. 2+3=5

(c) Show that every V_2 is an Einstein space. 5

10. (a) For a V_2 referred to an orthogonal system of parametric curves, show that $R_{12} = 0, R_{11}g_{22} = R_{22}g_{11} = R_{1221}$.

$R = g^{ij}R_{ij} = \dfrac{2R_{1221}}{g_{11}g_{22}}$, consequently $R_{ij} = \dfrac{1}{2}Rg_{ij}$ 6

(b) Define curl of a congruence and show that if a congruence satisfies two of the following conditions, it will satisfy the third: 8

(i) that it will be a normal congruence

(ii) that it be a geodesic congruence

(iii) that it be an irrotational