2022

M.Sc.

Third Semester

CORE – 10

MATHEMATICS

Course Code: MMAC 3.21

(Topology)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Show that if $\mathcal{B}$ be a basis for topology on X, then the collection $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for subspace topology on Y. 5

   (b) If τ and τ' are topologies on X and τ' is strictly finer than τ, what can you say about the corresponding subspace topologies on the subset Y of X? 5

   (c) Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ in X. Then, prove that the closure of A in Y equals $\overline{A} \cap Y$. 4

2. (a) Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. 5

   (b) Are lower limit topology and standard topology on ¡ comparable? Justify. 5

   (c) Show that a subspace of a Hausdorff topological space is Hausdorff. 4

UNIT–II

3. (a) Let X and X' denote a single set in the two topologies τ and τ', respectively. Let $i : X' \to X$ be the identity function.

   (i) Show that i is continuous $\Leftrightarrow$ τ' is finer than τ. 3

   (ii) Show that i is a homeomorphism $\Leftrightarrow$ $\tau = \tau'$. 3

(b) Define the quotient topology. Let $A = \{a_1, a_2, a_3\}$ and $p : ¡ \to A$ is defined as follows:

$$p(x) = \begin{cases} a_1 & \text{if } x > 0 \\ a_2 & \text{if } x < 0 \\ a_3 & \text{if } x = 0 \end{cases}$$

Obtain the quotient topology on A induced by p. 4

(c) Let $p : X \to Y$ be an open map and let A be an open subset of X. Prove that the map $q : A \to p(A)$ obtained by restricting p is also an open map. 4

4. (a) Show that the subspace (a, b) of ¡ is homeomorphic with $(0,1)$. 5

(b) Show that if $f : A \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ then f is continuous if and only if $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. 4

(c) Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \cup A_\alpha$. Let $g : X \to Y$; suppose that $g \mid A_\alpha$ is continuous for each α.

(i) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then g is continuous. 3

(ii) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but g is not continuous. 2

UNIT–III

5. (a) Prove that the image of a connected space under a continuous map is connected. 4

(b) Define a path in a topological space X. When is a space X called path connected? Show that a path connected topological space X is connected. 1+1+4=6

(c) Let $\{A_\alpha\}$ be a collection of connected subsets of X; let A be a connected subset of X. Show that if $A \cap A_\alpha \neq \varnothing$ for every α, then $A \cup (\cup A_\alpha)$ is connected. 4

6. (a) Define a linear continuum. Prove that if L is a linear continuum in the order topology, then L is connected. 2+5=7

(b) Is the space $(\mathbb{R}, \tau_l)$ connected? Justify your answer. 2

(c) If X is a topological space, each component of X lies in a component of X. If X is locally path connected, then prove that the components and the path components of X are the same. 5

UNIT–IV

7. (a) Prove that every compact subset of a Hausdorff topological space is closed. 5

(b) Show that a finite union of compact sets is compact. 4

(c) Let X be a compact topological space and Y an ordered set in the order topology. If $f : X \to Y$ is continuous, then show that there exists a point p such that $f(x) \leq f(p)$ for every $x \in X$. 5

8. (a) Prove that every closed interval in $\mathbb{R}$ is compact. 5

(b) Prove that the product of two compact spaces is compact. 5

(c) Show that the rationals $\mathbb{Q}$ are not locally compact. 4

UNIT–V

9. (a) Let X be a space satisfying the first countability axiom. Prove that the point x belongs to the closure $\overline{A}$ of the subset A of X if and only if there is a sequence of points of A converging to x. 5

(b) Show that if X is normal, every pair of disjoint closed sets have neighbourhoods whose closure is disjoint. 4

(c) Prove that every metrizable space is normal. 5

10. (a) Prove the following: 3½×2=7

(i) A subspace of a regular space is regular space.

(ii) A product of regular spaces is a regular space.

(b) Show that every locally compact Hausdorff space is regular. 4

(c) The topological space $\mathbb{R}_l$ is first countable but not second countable. Justify. 3