2022

M.Sc.

First Semester

CORE – 04

MATHEMATICS

Course Code: MMAC 1.41

(Abstract Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) If G is a group such that $(ab)^n = a^n b^n$ for 3 consecutive integers n, for all $a, b \in G$, show that G is abelian. Is the converse true? Justify. 5

   (b) Let G be a group of order n and let S be a subgroup of G. Determine the number of distinct right cosets of S in G. Justify. 4

   (c) Let G and H be groups and let $\phi : G \to H$ be a homomorphism. If $a \in G$ is of finite order, prove that $\phi(a) \in H$ is of finite order. Also, prove that the order of $\phi(a)$ divides the order of a. If $\phi(a) \in H$ is of finite order, does it imply that $a \in G$ is of finite order? Justify. 5

2. (a) Let $G = \langle g \rangle$ be a cyclic group of order n. For any $k \in \mathbb{N}$, prove that $O(g^k) = \dfrac{O(g)}{\gcd(n,k)}$. Determine how many elements of G have order n. 6

   (b) Let N be a subgroup of a group G. Prove that the following statements are equivalent:

   (i) N is a normal subgroup of G

   (ii) $gNg^{-1} = N$ for all $g \in G$

   (iii) Product of two right cosets of N in G is again a right coset of N in G. 3

(c) Show that every subgroup of an abelian group is normal. Is the converse true? Justify. 5

UNIT–II

3. (a) Let G be a group and H a subgroup of G. Let S denote the set of all right cosets of H in G. Can you define a group action of G on S? Justify. 5

(b) If $O(G)$ is pq where p and q are distinct prime numbers and if G has a normal subgroup of order p and a normal subgroup of order q, prove that G is cyclic. 5

(c) Find the conjugate of $(1 \;\; 2 \;\; \ldots \;\; n)$ in S_n. 4

4. (a) If p is a prime number and p divides $O(G)$, then prove that p has an element of order p. 7

(b) Prove that if P is the only Sylow p-subgroup of G, then P is normal in G. 4

(c) Prove that any group of order p^2 is abelian, where p is a prime. 3

UNIT–III

5. (a) Prove that any group of order 72 cannot be simple. 6

(b) Find the number of conjugates of $(1 \;\; 2)(3 \;\; 4)$ in S_n for $n \geq 4$. 4

(c) Let G be a group and $T = G \times G$. Show that

$D = \{(g, g) \in G \times G \mid g \in G\}$ is group isomorphic to G. 4

6. (a) Let G be a group and m be a positive integer such that m divides $O(G)$. Does there always exist a subgroup of order m? Justify. 6

(b) Let H be a subgroup of a group G. Prove that the number of conjugates of H in G is $O\left(\dfrac{G}{N(H)}\right)$ where $N(H)$ denotes the normalizer of H. 4

(c) Show that there is no simple group of order pqr, where p, q and r are distinct primes. 4

UNIT–IV

7. (a) Let R be a ring such that $x^2 = x$ for all $x \in R$. Prove that R is a commutative ring. Show that the converse need not hold. 4

(b) Let Z be the ring of integers and let I be an ideal of Z. Show that $I = \{na \mid n \in \mathbb{Z}\}$ for a fixed $a \in \mathbb{Z}$, that is, show that I is the set of all integer multiples of a fixed integer a. What values of a is necessary and sufficient for I to be maximal in Z? Justify. 6

(c) Let $\phi : R \to R'$ be a ring homomorphism. If R is a field, prove that either ϕ is an isomorphism or ϕ maps every element of R to the additive identity (zero element) of R'. 4

8. (a) Let R be an integral domain and let the characteristic of R be a finite $n \in \mathbb{N}$. Show that n is a prime number. Provide an example of an integral domain which has an infinite number of elements but has finite characteristic. 3

(b) Let R be a commutative ring with unity and let I be an ideal of R. Prove that I is a prime ideal if and only if $\frac{R}{I}$ is an integral domain. 4

(c) Prove that every PID is a UFD. 7

UNIT–V

9. (a) If the primitive polynomial $f(x)$ can be factored as the product of two polynomials having rational coefficients, prove that $f(x)$ can be factored as the product of two primitive polynomials. Does the result hold for $f(x)$ not primitive? Justify. 6

(b) Let R be an integral domain (with unity) and let $f(x),\ g(x) \in R[x]$. Prove that $\deg\{f(x)g(x)\} = \deg f(x) + \deg g(x)$. Does the result hold if R is a commutative ring with unity but not an integral domain? Justify. 5

(c) Show that the polynomial $f(x) = 2x^2 + 4$ is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$. 3

10. (a) If R is an integral domain, show that $R[x]$ is also an integral domain. If R is a field, is $R[x]$ also a field? Justify. If R is a commutative ring with unity, is $R[x]$ also a commutative ring with unity? Justify. 7

(b) Let F be a field and $f(x),\ g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exist unique polynomials $q(x),\ r(x) \in F[x]$ such that $f(x) = g(x)\, q(x) + r(x)$, where either $r(x) = 0$ or deg $r(x) <$ deg $g(x)$. 5

(c) If p is a prime number, prove that the polynomial $x^n - p$ is irreducible over $\mathbb{Q}$. 2