2022

M.Sc.

First Semester

CORE – 03

MATHEMATICS

Course Code: MMAC 1.31

(Real Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Let (X, d) be a metric space. Show that a subset A of X is open if and only if it is the union of open balls in A. 5

   (b) Prove that the Cartesian product of two countable sets is countable. 5

   (c) Prove/disprove: Sets A and B intersect if and only if $d(A, B) = 0$. 4

2. (a) Is ¡ compact? Justify. 5

   (b) Prove/disprove: $\left\{\frac{1}{n} \middle| n \in ¥\right\}$ is open in $(¡\ , d)$, where d is the usual metric space. 3

   (c) Prove $c = 2^{\aleph_0}$, where c is the cardinality of ¡ and $\aleph_0$ is the cardinality of a countable set. 6

UNIT–II

3. (a) Let (x_n) be a sequence in a metric space (X, d) and let $x \in X$. If (x_n) converges to x and (x_n) has infinitely many distinct values, then prove that x is a limit point of the range of (x_n). 5

   (b) Prove/disprove: The series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$ is convergent. 3

(c) If $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0,1,2,\ldots$, then prove that $\sum_{n=0}^{\infty} c_n = AB$. 6

4. (a) Let (X,d) be a metric space, $A \subseteq X, x \in A'$. Show that there exists a sequence (x_n) in *A* which converges to *x*. 5

(b) When is $\sum \frac{1}{n^p}$ convergent? Justify. 5

(c) Show that the product of two convergent series may diverge. 4

UNIT–III

5. (a) For metric spaces *X* and *Y*, prove that a function *f* is continuous from *X* to *Y* if and only if for every open set *E* in *Y*, $f^{\leftarrow}(E)$ is open in *X*. 7

(b) State and prove the generalised mean value theorem. 7

6. (a) Prove/disprove: If *X* is compact and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is compact. 7

(b) Give an example each of the following: 7

(i) A function $f : A \to B$ which is bijective such that *f* is differentiable on *A* but f^{-1} is not differentiable on *B*.

(ii) A function *f* which is differentiable at a point *x* but f' is not differentiable at *x*.

UNIT–IV

7. (a) $f : [a,b] \to \mathbb{R}$ is bounded and *P*, *P** are partitions of $[a,b]$ such that *P** is finer than *P*. Prove that $L(P,f) \le L(P^*,f)$. 7

(b) If *f* is continuous on $[a,b] \subset \mathbb{R}$ and *f* is real-valued, prove that *f* is integrable. Is the converse true? Justify. 7

8. (a) State and prove a criterion of integrability. 7

(b) If f and g are integrable on $[a,b] \subset \mathbb{R}$, then prove that $f+g$ and fg are also integrable. 7

UNIT–V

9. (a) State and prove the Cauchy criterion for uniform convergence. 7

(b) Let $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in \mathbb{R}, n \in \mathbb{N}$

Discuss the convergence of (f_n) and (f_n'). 7

10. (a) Discuss the convergence of $\left(\dfrac{nx}{1+n^2x^2}\right)$ on the intervals $[a,\infty[$ and $[0,\infty[$, where $a > 0$. 7

(b) Let (f_n) and (g_n) be sequences of bounded functions on A that converge uniformly on A to f and g respectively. Show that $(f_n g_n)$ converges uniformly on A to fg. 7