2022
M.Sc.
First Semester
CORE – 02
MATHEMATICS
Course Code: MAC 1.21
(Linear Algebra)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Determine whether the following subsets of $\mathbf{F}^3$ are subspaces of $\mathbf{F}^3$: 2+2=4

   (i) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

   (ii) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$.

   (b) Prove that a list $v_1, v_2, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$, where $a_1, a_2, \ldots, a_n \in \mathbf{F}$. 5

   (c) Suppose $v_1, v_2, \ldots, v_n$ is a basis if V and $w_1, w_2, \ldots, w_n$ is a basis of W. Prove that $\mathcal{L}(V, W)$ is isomorphic to $\mathbf{F}^{m,n}$ 5

2. (a) Let $T \in \mathcal{L}(V, W)$. Prove that range (T) is subspace of W. 4

   (b) Prove that the intersection of any finite collection of subspace of V is a subspace of V. 5

   (c) Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible. 5

UNIT–II

3. (a) Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. Find a basis of $\mathbf{F}^3$ with respect to which T has a diagonal matrix. 4

(b) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T, then prove that
$E(\lambda_1, T) + E(\lambda_2, T) + \ldots + E(\lambda_m, T)$ is a direct sum. 5

(c) Find the minimal polynomial of the operator $T \in \mathcal{L}(\pounds^{3})$ defined by
$T(z_1, z_{2,} z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3)$ 5

4. (a) Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_{2,} z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T. 4

(b) Prove that every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue. 5

(c) Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$. Prove that $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.
5

UNIT–III

5. (a) Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null$(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$. 4

(b) Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$, then prove that there is a basis of V that is a Jordan basis for T. 5

(c) Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let q denote the characteristic polynomial of T. Then show that $q(T) = 0$. 5

6. (a) Suppose U is a finite-dimensional subspace of V and P_U denote the orthogonal projection of V onto U. Show that P_U is a linear map on V. 6

(b) Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an upper triangular matrix with some basis of V. 8

UNIT–IV

7. (a) If U is a subset of V, then prove that the orthogonal complement of U is a subspace of V. 4

(b) Prove that $\|u+v\| \leq \|u\|+\|v\|$, for $u, v \in V$. Also, show that the inequality is an equality if and only if one of *u*, *v* is a non-negative multiple of the other. 5

(c) Show that if *V* is a real inner-product space, then the set of self-adjoint operators on *V* is a subspace of $\mathcal{L}(V)$. 5

8. (a) Prove that $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$ for $u, v \in V$. 4

(b) On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x) q(x)\, dx$$

Apply the Gram-Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$. 5

(c) If $T \in \mathcal{L}(V, W)$, then prove that the adjoint of $T, T^* \in \mathcal{L}(V, W)$. 5

UNIT–V

9. (a) For any vector space *V*, prove that the sum of two bilinear forms and the product of a scalar and a bilinear form on *V* are again bilinear forms on *V*. 7

(b) Let *V* be a vector space of dimension *n* over **F**. Then prove that there is a one-to-one correspondence between the set of bilinear forms on *V* and the set of $n \times n$ matrices over **F**. 7

10. (a) Define a function $H : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $H\left((a_1, a_2), (b_1, b_2)\right) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2$ for $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$. Verify that *H* is a bilinear form on $\mathbb{R}^2$. 5

(b) Determine the matrix of *H* defined above with respect to the ordered basis $(1,1), (1,-1)$ and also with respect to the standard basis of $\mathbb{R}^2$. 4

(c) Let *H* be a hermitian form on *V*. Then prove that a linear mapping *T* of *V* into itself is *H*-unitary if and only if $H\left(T(x), T(x)\right) = H(x, x)$ for all $x \in V$. 5
