2022
M.Sc.
First Semester
CORE – 01
MATHEMATICS
Course Code: MMAC 1.11
(Ordinary Differential Equations)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Let ϕ be any solution of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0. Then for all x in I prove that $\|\phi(x)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x)\| e^{k|x-x_0|}$ where $\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{\frac{1}{2}}$, $k = 1 + |a_1| + |a_2|$. 6

   (b) Consider the constant coefficient equation $L(y) = y'' + a_1 y' + a_2 y = 0$. Let ϕ_1 be the solution satisfying $\phi_1(x_0) = 1$, $\phi_1'(x_0) = 0$, and let ϕ_2 be the solution satisfying $\phi_2(x_0) = 0$, $\phi_2'(x_0) = 1$. And let ϕ be the solution satisfying $\phi(x_0) = \alpha$, $\phi'(x_0) = \beta$, than show that $\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x)$ for all x. 4

   (c) (i) Show that the function ϕ_1, ϕ_2 defined by $\phi_1(x) = x^2$, $\phi_2(x) = x|x|$ are linearly independent for $-\infty < x < \infty$. 2

   (ii) Compete the Wronskian of these functions. 2

2. (a) If ϕ_1, ϕ_2 are two solutions of $L(y) = 0$ on an interval I containing a point x_0, then show that $W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2)(x_0)$. 4

(b) Determine all complex numbers l for which the problem $-y'' = ly$, $y(0)=0$, $y(1)=0$ has a non-trivial solution, and compute such solution for each of these l. 5

(c) Find all the solution of $y'' + y = 2\sin x \sin 2x$. 5

UNIT–II

3. (a) Find two linearly independent solutions of the equation

$$(3x-1)^2 y'' + 4(9x-3)y' - 36y = 0 \text{ for } x > \frac{1}{3}.$$ 4

(b) If $\phi_1, \phi_2, \phi_3 \ldots \phi_n$ are n solutions of $L(y)=0$ on an interval I, then prove that they are linearly independent if and only if $W(\phi_1 \cdots \phi_n)(x) \neq 0$ for all x in I. 5

(c) Let ϕ be real valued non-trivial solution of $y'' + \alpha(x)y = 0$ on $a<x<b$ and let ψ be a real valued solution of $y'' + \beta(x)y = 0$ on $a<x<b$. Here α, β are real valued continuous functions. Suppose that $\beta(x) > \alpha(x), (a<x<b)$. Then show that if x_1 and x_2 are successive zeros of ϕ on $a<x<b$, then ψ must vanish at some point ξ, $x_1 < \xi < x_2$. 5

4. (a) State and prove the uniqueness theorem related to initial value problem of linear equation with variable coefficients for the homogeneous equations. 4

(b) Let $\phi_1, \phi_2 \ldots \phi_n$ be n solutions of $L(y)=0$ on the interval I, and let x_0 be any point in I. Then show that

$$W(\phi_1, \phi_2 \cdots \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\phi_1, \phi_2 \cdots \phi_n)(x_0)$$ 6

(c) Consider the equation $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ where a_1, a_2 are continuous on some interval I. Show that a_1, a_2 are uniquely determined by any basis ϕ_1, ϕ_2 for the solution of $L(y)=0$. 4

UNIT–III

5. (a) Find all the solutions of the equation $y'' - \frac{2}{x^2} = x$, $(0<x<\infty)$. 6

(b) Find the solution of $y'' + (x-1)^2 y' - (x-1)y = 0$ in the form $\phi(x) = \sum_{k=0}^{\infty} c_k (x-1)^k$, which satisfies $\phi(1) = 1$, $\phi'(1) = 0$. 8

6. (a) Show that there are constants $\alpha_0, \alpha_1 \cdots \alpha_n$ such that $x^n = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \ldots \alpha_n P_n(x)$, where $P_n(x)$ is the n^{th} Legendre polynomial. 5

(b) If $P_n(x)$ is n^{th} Legendre polynomial, then prove that

$$\int_{-1}^{1} P_n^2(x) = \frac{2}{(2n+1)}.$$ 5

(c) Verify that the function Q_1 defined by $Q_1(x) = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1$, $(|x| < 1)$ is a solution of the Legendre equation where $\alpha = 1$. 4

UNIT–IV

7. (a) Let M, N be two real valued function which has continuous first order partial derivatives on some rectangle $R : |x - x_0| \le a$, $|y - y_0| \le b$. Then show that the equation $M(x, y) + N(x, y)y' = 0$ is exact in *R* if, and only if, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in *R*. 6

(b) Find any integrating factor of the following equation and solve it, $\cos x \cos y \, dx - 2\sin x \sin y \, dy = 0$ 4

(c) Consider the problem $y' = 1 + y^2$, $y(0) = 0$. 4

(i) Using the separation of variables, find the solution of ϕ of this problem.

(ii) Show that all the successive approximations $\phi_o, \phi_1, \phi_2 \ldots$ exist for all real *x*.

(iii) Show that $\phi_k(x) \to \phi(x)$ for each *x* satisfying $|x| \le \frac{1}{2}$.

8. (a) Consider the equation $M(x,y)dx + N(x,y)dy = 0$, where M, N have continuous first order partial derivatives of some rectangle R. Show that 5

(i) the given equation has an integrating factor u, which is a function of x alone, then $p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a continuous function of x alone.

(ii) If p is continuous and independent of y then $u(x) = e^{P(x)}$, where $P' = p$.

(b) Show that the successive approximation defined by

$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))dt, \phi_0(x) = y_0$ exist as continuous function on $I : |x - x_0| \le a = \min\left[a, \frac{b}{M}\right]$, and $(x, \phi_k(x))$ is in R for x in I. And ϕ_k satisfy $|\phi_k(x) - y_0| \le M|x - x_0|$ for all x in I. 5

(c) Consider the equation $y' = f(x)p(\cos y) + g(x)q(\sin y)$ where f, g are continuous for all real x, and p, q are polynomials. Show that every initial value problem for this equation has a solution which exist for all real x. 4

UNIT–V

9. (a) Find all the solution of the equation $x^2y'' - 5xy' + 9y = x^2$ for $x > 0$. 7

(b) Find the singular point and compute the indicial polynomial for all equation $x^2y'' + (\sin x)y' + (\cos x)y = 0$. 7

10. (a) Solve the following equation in series
$2x^2y'' - xy' + (1 - x^2)y = 0$. 10

(b) Show that $x = \infty$ is a regular singular point of
$x^2y'' + 4xy' + 2y = 0$. 4