2022

B.A./B.Sc.

Fifth Semester

CORE – 12

MATHEMATICS

Course Code: MAC 5.21

(Group Theory-II)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Prove that the set of all inner automorphisms of G is a subgroup of automorphism of G. 4

   (b) Let G' be the commutator subgroup of a group G then prove that

   (i) G' is normal in G

   (ii) $\frac{G}{G'}$ is abelian 2+2=4

   (c) Define characteristic subgroup of a group G. If H is a characteristic subgroup of K and K is normal in G, show that H is normal in G. 1+5=6

2. (a) Prove that $\frac{G}{Z(G)} \cong Inn(G)$, where $Z(G)$ is the centre of G and $Inn(G)$ is the set of all inner automorphisms of G. 5

   (b) If G is a group, prove that the set of all automorphisms of G is also a group. 5

   (c) Find Aut$(¢_{10})$. 4

UNIT–II

3. (a) Find the elements of order 9 in $¢_3 \oplus ¢_9$. Also, find the order of the largest cyclic subgroup of $¢_3 \oplus ¢_9$. 3+2=5

(b) Find all non-isomorphic abelian groups of order 360. 4

(c) Prove that ¢ $_m \oplus$ ¢ $_n \cong$ ¢ $_{mn}$ if and only if gcd(m, n)=1. 5

4. (a) Prove that ¢ $_8 \oplus$ ¢ $_2$ is not isomorphic to ¢ $_4 \oplus$ ¢ $_4$. 5

(b) How many Abelian groups (up to isomorphism) are there 2+2=4

(i) of order 6?

(ii) of order 24?

(c) Prove that every group of order p^2, where p is a prime, is isomorphic to ¢ $_{p^2}$ or ¢ $_p \oplus$ ¢ $_p$. 5

UNIT–III

5. (a) Define action of a group G on a set A. Show that the additive group ¢ acts on itself by $z.a = z + a$ for all $z, a \in$ ¢ . 1+3=4

(b) State and prove generalized Cayley's theorem. 1+4=5

(c) Let G be a group of order n and p be the smallest prime dividing $|G|$. Then prove that any subgroup of index p is normal. 5

6. (a) Let H be a subgroup of a finite group G and let H act on G by left multiplication. Let $x \in G$ and let O be the orbit of x under the action of H. Prove that the map $H \to O$ defined by $h \mapsto hx$ is a bijection. From this, deduce Lagrange's theorem for the subgroup H of the group G. 3+2=5

(b) Let $G = D_4$ act on the set A consisting of the four vertices of a square. If $a \in A$, then find the stabilizer of a and the kernel of this action. Is the action of G on A faithful? 2+2=4

(c) Let G be a group acting on a nonempty set A. Prove that the relation on A defined by $a : b$ if and only if $a = g \cdot b$ for some $g \in G$ is an equivalence relation. 5

UNIT–IV

7. (a) Define p-group and prove that a finite group G is a p-group if and only if $|G| = p^n$. 4

   (b) Define conjugacy classes of a group. Determine whether $\sigma_1 = (1\ 2\ 3)(4\ 5\ 6\ 7)$ and $\sigma_2 = (3\ 5\ 7)(1\ 4\ 2\ 6)$ are conjugate in S_7. If so, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$. 1+2+2=5

   (c) Let G be a finite group and $g_1, g_2, ...g_r$ be representatives of the distinct conjugacy classes of G not contained in the centre $Z(G)$.

   Then prove that $|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$. 5

8. (a) Determine the conjugacy classes of elements of A_4 and verify the class equation. 2+2=4

   (b) Prove that the number of conjugates of an element x in G is the index of the centralizer of x, $|G : C_G(x)|$. 5

   (c) Prove that two elements in S_n are conjugate in S_n if and only if they have the same cycle type. 5

UNIT–V

9. (a) Prove that any two Sylow p-subgroups of a group G are conjugate in G. 4

   (b) Let $|G| = 30$. Show that 2+2+2=6

   (i) Either Sylow 3-subgroup or Sylow 5-subgroup is normal in G.
   (ii) G has a normal subgroup of order 15.
   (iii) Both Sylow 3-subgroup and Sylow 5-subgroup are normal in G.

   (c) Prove that there is no simple group of order 210. 4

10. (a) Let $|G| = pq$, where p, q are distinct primes, $p < q, p \nmid q-1$. Show that G is cyclic. 5

(b) Let G be a group of order 231. Show that 11-Sylow subgroup of G is contained in the centre of G. 5

(c) Prove that a Sylow p-subgroup of a finite group G is a normal subgroup of G if and only if it is the only Sylow p-subgroup of G. 4