**2022**

**B.A./B.Sc.**

**Third Semester**

CORE – 6

**MATHEMATICS**

*Course Code: MAC 3.21*

(Group Theory – I)

*Total Mark: 70* *Pass Mark: 28*

*Time: 3 hours*

*Answer* five *questions, taking* one *from each unit.*

**UNIT–I**

1. (a) Show that in a group $G$, there is only one identity element. 2

   (b) Show that $\{1,2,3\}$ under multiplication modulo 4 is not a group. 2

   (c) Prove that the set of all $2\times2$ matrices with real entries and determinant 1 is a group under matrix multiplication. 5

   (d) Prove that if $(ab)^2 = a^2b^2$ in a group $G$, then $ab = ba$. 5

2. (a) Show that for group elements $a$ and $b$, $(ab)^{-1} = b^{-1}a^{-1}$. 2

   (b) Give an example of group elements $a$ and $b$ with the property that $a^{-1}ba = b$. 2

   (c) Prove that if $G$ is a group with the property that square of every element is the identity, then $G$ is abelian. 5

   (d) Describe in pictures or words the elements of $D_4$. 5

**UNIT–II**

3. (a) What do you mean by order of an element in a group? Give an example. 2

   (b) Let $< a >$ denote the set $\{a^n \mid n \in \mathbb{Z}\}$. Show that for any element $a$ in $G$, $< a >$ is a subgroup of $G$. 2

   (c) Suppose that a cyclic group $G$ has exactly three subgroups: $G$ itself, $\{e\}$, and a subgroup of order 7. What is the order of $G$? What can you say if 7 is replaced with $p$ where $p$ is a prime? 2+3=5

(d) Prove that for each $a$ in group $G$, the centralizer of $a$ is a subgroup of $G$. 5

4. (a) Find the centres of the dihedral group, $D_n$, where $n$ is even. 2

(b) Show that $H = \{x^2 \mid x \in G\}$ is a subgroup of the abelian group $G$. 2

(c) Let $G$ be a group and $H$ be a subgroup of $G$. Show that normalizer of $H$ is a subgroup of $G$. 5

(d) Prove or disprove: Any group of prime order $p$ is always cyclic. 5

## UNIT–III

5. (a) Let $\alpha$ and $\beta$ belong to $S_n$. Prove that $\alpha\beta$ is even if and only if $\alpha$ and $\beta$ are both even or both odd. 2

(b) Give two reasons why the set of odd permutations of $S_n$ is not a subgroup. 2

(c) Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 5

(d) Define alternating group. Prove that for $n > 1$, $A_n$ has order $n!/2$. 5

6. (a) Let $\alpha$ and $\beta$ belong to $S_n$. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation. 2

(b) Show that $A_8$ contains an element of order 15. 2

(c) If the pair of cycles $\alpha = (a_1, a_2, ... a_m)$ and $\beta = (b_1, b_2, ... b_n)$ have no entries in common, then show that $\alpha\beta = \beta\alpha$. 5

(d) Prove that every permutation in $S_n$, $n > 1$, is a product of 2-cycles. 5

## UNIT–IV

7. (a) Show that $A_n$ is a normal subgroup of $S_n$. 2

(b) Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has seven subgroups of order 2. 2

(c) Let $G$ and $H$ be cyclic group. Then prove that $G \oplus H$ is also cyclic if and only if $|G|$ and $|H|$ are relatively prime. 5

(d) State and prove Cauchy's theorem for abelian groups. 1+4=5

8. (a) Prove that if $H$ has index 2 in $G$, then $H$ is a normal subgroup of $G$. 2

(b) In $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$, find two subgroups of order 12. 2

(c) Prove or disprove: $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group. 5

(d) Let $G$ be a group and $\mathbb{Z}(G)$ be the centre of $G$. Prove that if $G / \mathbb{Z}(G)$ is cyclic, then $G$ is abelian. 5

## UNIT–V

9. (a) Let $\phi$ be a group homomorphism from $G$ to $\overline{G}$. Then show that $Ker\ \phi$ is a normal subgroup of $G$. 2

(b) Define automorphism of a group. Let $G$ be a finite abelian group and $G$ has no element of order 2. Show that the mapping $g \to g^2$ is an automorphism of $G$. 1+1=2

(c) State and prove the first theorem of isomorphism. 1+4=5

(d) If $\phi$ is a homomorphism from $G$ to $H$ and $\sigma$ is a homomorphism from $H$ to $K$, show that $\sigma\phi$ is a homomorphism from $G$ to $K$. How are $Ker\ \phi$ and $Ker\ \sigma\phi$ related? 5

10. (a) Show that $\mathbb{Z}_n / < k > \cong \mathbb{Z}_k$, where $k$ is a divisor of $n$. 2

(b) Suppose that $\phi$ is an isomorphism from a group $G$ onto a group $\overline{G}$. Then prove the following: 1×2=2

(i) $\phi$ carries identity of $G$ to the identity of $\overline{G}$.

(ii) For any elements $a$ and $b$ in $G$, $a$ and $b$ commute $\Leftrightarrow$ $\phi(a)$ and $\phi(b)$ commute.

(c) If $M$ and $N$ are normal subgroups of $G$ and $N \leq M$, then prove that $(G/N)/(M/N) \cong G/M$. 5

(d) Prove that every group is isomorphic to a group of permutations. 5