2022

B.A./B.Sc.

Third Semester

CORE – 5

MATHEMATICS

Course Code: MAC 3.11

(Theory of Real Functions)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Establish the limit of $\lim\limits_{x\to 0}\frac{x^2}{|x|}=0$ using the $\varepsilon-\delta$ definition of limit. 4

   (b) State and prove the sequential criterion for limits. 1+4=5

   (c) Find functions *f* and *g* defined on $(0,\infty)$ such that $\lim\limits_{x\to\infty} f=\infty$ and $\lim\limits_{x\to\infty} g=\infty$ and $\lim\limits_{x\to\infty}(f-g)=0$. Can you find such functions, with $g(x)=0$ for every $x\in(0,\infty)$, such that $\lim\limits_{x\to\infty}\frac{f}{g}=0$. 5

2. (a) Determine whether $\lim\limits_{x\to 0}\sin\frac{1}{x^2}=\infty$ exist in ¡ . 4

   (b) Let $A\subseteq\mathbb{R}$, let f: $A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. If $\lim\limits_{x\to c} f$ exists, and if $|f|$ denotes the function defined for $x\in A$ by $|f|(x):=|f(x)|$, prove that $\lim\limits_{x\to c}|f|=\left|\lim\limits_{x\to c} f\right|$. 5

   (c) Suppose that *f* and *g* have limits in $\mathbb{R}$ as $x\to\infty$ and that $f(x)\le g(x)\forall x\in(a,\infty)$. Prove that $\lim\limits_{x\to\infty} f\le\lim\limits_{x\to\infty} g$. 5

UNIT–II

3. (a) Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $\left|f(x) - f(y)\right| \le K\left|x - y\right| \forall x, y \in \mathbb{R}$. Show that *f* is continuous at every point $c \in \mathbb{R}$. 4

(b) Show that if a function $f : I \to \mathbb{R}$ is continuous on *I* which is closed and bounded interval then the image of set $f(I)$ is also closed bounded interval. Illustrate the same will not hold true for an open interval. 5

(c) Show that the function $f(x) := \dfrac{1}{x^2}$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := [0, \infty)$. 5

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at *c* and let $f(c) > 0$. Show that there exists a neighbourhood $V_\delta(c)$ of *c* such that if $x \in V_\delta(c)$, then $f(x) > 0$. 4

(b) State and prove Bolzano's intermediate value theorem. 1+4=5

(c) Show that every polynomial of odd degree with real coefficients has at least one real root. 5

UNIT–III

5. (a) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) := x^2 \sin\left(\dfrac{1}{x^2}\right)$ for $x \neq 0$ and $h(0) : 0$. Show that h is differentiable for every $x \in \mathbb{R}$. Also show that the derivative h' is not bounded on the interval [–1,1]. 5

(b) Let *I* be an interval and let $h : I \to \mathbb{R}$ be differentiable on *I*. Show that if h' is positive on *I*, then *h* is strictly increasing on *I*. 4

(c) Let $I \subseteq \mathbb{R}$ be an interval, $c \in I, f, g : I \to \mathbb{R}$ be functions that are differentiable at c. If $g(c) \neq 0$, then show that function $\frac{f}{g}$ is differentiable at c and hence find its derivative. 5

6. (a) Determine where each of the following functions from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative: 2+1+1=4

(i) $f(x) := |x| + |x+1|$

(ii) $g(x) := x|x|$

(b) State and prove the interior extremum theorem. 1+4=5

(c) State the mean value theorem. Apply the theorem to prove that $|\sin x| \leq x$ for every $x \geq 0$. 1+4=5

UNIT–IV

7. (a) State the intermediate value property of derivatives. 2

(b) Let $f : [0,2] \to \mathbb{R}$ be continuous on [0,2] and differentiable on (0,2) with $f(0)=0, f(1)=1, f(2)=1$. Show that: 2×3=6

(i) $\exists c_1 \in (0,1)$, such that $f'(c_1) = 1$

(ii) $\exists c_2 \in (1,2)$, such that $f'(c_2) = 0$

(iii) $\exists c \in (0,2)$, such that $f'(c) = 1$

(c) Evaluate 3×2=6

(i) $\lim\limits_{x \to 0} \frac{1}{x(\ln x)^2}$ in (0,1) (ii) $\lim\limits_{x \to \infty} \left(1 + \frac{3}{x}\right)^x$ in $(0, \infty)$

8. (a) Let I be an interval and let $g : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative g' is never 0 on I, then either $g'(x) > 0$ for every $x \in I$ or $g'(x) < 0$ for every $x \in I$. 4

(b) If $\alpha > 1$, then prove that $(1+x)^\alpha \geq 1 + 2\alpha x$ for every $x > -1$, equality holds if and only if $x = 0$. 5

(c) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for every $x \geq 0$. Show that $f(x) \leq g(x)$ for every $x \geq 0$. 5

UNIT–V

9. (a) State the Taylor's theorem. What is the importance of Taylor's theorem? What is the use of the remainder term in the theorem? 4

(b) Determine whether or not $x = 0$ in a point of relative extremum of the functions: 5

(i) $f(x) = \sin x + \frac{1}{6}x^3$ (ii) $g(x) = \cos x - 1 + \frac{1}{2}x^2$

(c) If $x \in [0,1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$ Use this to approximate ln 1.5 with an error less than 0.001. 5

10. (a) Prove that $e = 1 + \frac{!}{1!} + \frac{1}{2!} + \frac{!}{3!} + \ldots$ 4

(b) If $x > 0$, show that $\left| (1+x)^{\frac{1}{3}} - (1 + \frac{1}{3}x - \frac{1}{9}x^2) \right| \leq \left(\frac{5}{81} \right) x^3$. Use this inequality to approximate $\sqrt[3]{1.2}$ and $\sqrt[3]{2}$. 5

(c) Show that $\left| \sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right) \right| < \frac{1}{5040}$ for $|x| \leq 1$. 5