2022

B.A./B.Sc.

First Semester

CORE – 2

MATHEMATICS

Course Code: MAC 1.21

(Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Find $|z|$, Arg z, arg z, arg $\bar{z}$ for $z=(1-i)(6+6i)$ 1+1+1+1=4

(b) Find the number of ordered pairs (a,b) of real numbers such that $(a+bi)^{2022}=a-bi$ 5

(c) Compute $z=\dfrac{(1-i)^{10}\left(\sqrt{3}+1\right)^{5}}{\left(-1-i\sqrt{3}\right)^{10}}$ 5

2. (a) Find the cube roots of $z=1-i$ and represent them in the complex plane. 4

(b) Find the polar representation of the complex number $z=1+\cos\alpha+i\sin\alpha$. 5

(c) State and prove de Moivre's theorem. 5

UNIT–II

3. (a) For $a,b\in$ ¥ , define $a\sim b$ if and only if a^2+b is even. Show that $\sim$ is an equivalence relation on ¥ . 4

(b) If $ac\equiv bc \pmod{n}$ and $\gcd(c,n)=1$, then prove that $a\equiv b \pmod{n}$. 5

(c) Prove that $3^{2n}-1$ is divisible by 8 for every $n\in$ ¥ by induction. 5

4. (a) Show that composition of functions is associative. 4

(b) Prove that the intervals $(0,1)$ and $(10,\infty)$ have the same cardinality. 5

(c) Prove that the greatest common divisor of integers a and b is an integral combination of them. Hence, show that if $a|bc$ for $a,b,c \in ¢$ with $\gcd(a,b)=1$, then $a|c$. 5

UNIT–III

5. (a) Find the general solution of the linear system whose augmented matrix is 4

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

(b) Describe all the solutions of the following system in parametric vector form: 5

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

(c) Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E and 30% to M and retains the rest. Construct the exchange table for this economy and find a set of equilibrium price for the economy. 5

6. (a) For what values of h will $\mathbf{y}$ be in the Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if 4

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

(b) Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$NaHCO_3 + H_3C_6H_5O_7 \longrightarrow Na_3C_6H_5O_7 + H_2O + CO_2$

Balance the above chemical equation using vector equation. 5

(c) If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then prove that: 2½×2=5

(i) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(ii) $A(c\mathbf{u}) = c(A\mathbf{u})$

UNIT–IV

7. (a) Determine if the columns of the following matrix form a linearly independent set. Justify your answer. 4

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

(b) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^m$, is called an affine transformation. Is T a linear transformation? Justify. 5

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation with standard matrix A. Prove that T is one-to-one if and only if the column if A are linearly independent. 5

8. (a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that T is a one-to-one if and only if the equation $T(\mathbf{x}) = 0$ has only the trivial solution. 4

(b) Prove that the following statements are equivalent for an $n \times n$ matrix A: 5

(i) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $b \in \mathbb{R}^n$

(ii) The column of A span $\mathbb{R}^n$

(c) Find the inverse of the following matrix, using elementary row operations: 5

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

UNIT–V

9. (a) Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is $\mathbf{u}$ in Nul A?
Is $\mathbf{u}$ in Col A? Justify your answers. 2+2=4

(b) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of a $n \times$ n matrix A, then prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ id linearly independent. 5

(c) Determine the eigenvalues and find the characteristic equation of

$$A = \begin{bmatrix} 5 & -5 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ 5

10. (a) Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$ 4

(b) Prove that the null space of an $m \times n$ matrix A is a subspace of ¡ ^{n}.
5

(c) Let $A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix}$. An eigenvalue of A is –4. Find a basis for the corresponding eigenspace. 5