May 2025
M.Sc.
Fourth Semester
CORE – 12
MATHEMATICS
Course Code: MMAC 4.21
(Rings & Modules)

Total Mark: 70   *Pass Mark: 28*
Time: 3 hours

Answer <u>five</u> questions, taking <u>one</u> from each unit.

UNIT–I

1. (a) Prove that every field is a division ring. What can you say about the converse?   4
   (b) If R is any ring, prove that the set $M_2(Q)$ of all matrices of order 2 with entries in Q is a ring under matrix addition and matrix multiplication.   5
   (c) If R is a ring, define R^{op} and prove that it is a ring.   5

2. (a) If G is an additive abelian group, prove that the set of all group endomorphisms is a ring under pointwise addition and composition of maps.   5
   (b) Define local ring and give some examples of local rings. Prove that the characteristic of a local ring is either 0 or a power of a prime.   4
   (c) If A and B are ideals of a ring R, prove that $A + B$ and AB are also ideals of R.   5

UNIT–II

3. (a) If R is a ring, prove that M is a left R-module if and only if M is a right R^{op}-module, where R^{op} is the ring opposite to R.   5
   (b) If R is a ring and N is a submodule of an R-module M, prove that the quotient group M/N is an R-module.   5

(c) If M and N are R-modules and $f : M \to N$ is an R-linear map, prove that the kernel of f is a submodule of M and image of f is a submodule of N. 4

4. (a) If M is an abelian group, R is a ring, and $End_Z(M)$ is the ring of all additive endomorphisms of M, prove that M is a right R-module if and only if there exists an anti-homomorphism $\psi' : R \to End_Z(M)$. 5

(b) If A and B are submodules of an R-module M, prove that
$$\frac{A+B}{A} \approx \frac{B}{B \cap A}.$$
5

(c) If A and B are submodules of an R-module M, prove that $A + B$ is a submodule of M, containing both A and B. 4

UNIT–III

5. (a) Define:
   (i) direct sum of two modules
   (ii) direct summand of a module
   If M and N are submodules of a module P over a ring R, prove that $M \cap N = (0)$ if and only if every element $t \in M + N$ can be uniquely written as $t = p + q$, where $p \in M$, $q \in N$. 4

(b) Define free module and prove that every vector space is a free module. 5

(c) Prove that every free module is a projective module and justify that the converse is not true by an example. 5

6. (a) Define a torsion module and a torsion-free module. If R is an integral domain and M is an R-module, show that the set T of the torsion elements of M is a submodule of M and the quotient module M / T is torsion free. 5

(b) Prove that the function $Hom_R(P,)$ is left exact and deduce the condition for P to be a projective module. 6

(c) Prove that any abelian group which has a non-trivial element of finite order cannot be free. 3

UNIT–IV

7. (a) Define simple module and prove that if A and B are simple modules over a ring R, then any R-linear map $f : A \rightarrow B$, is either the zero map or an isomorphism.   4

   (b) If $\{M_i\}_{i \in I}$ is a family of R-modules, prove that $\prod_{i \in I} M_i$ is injective if and only if each M_i is injective.   6

   (c) Define divisible module and give an example. If M is a divisible R-module and N is a submodule of M, prove that M / N is a divisible module.   4

8. (a) Prove that an R-module Q is injective if and only if every exact sequence of the form $0 \rightarrow Q \rightarrow M \rightarrow M'' \rightarrow 0$ splits.   6

   (b) Prove that $\mathbb{Z}$-module can be embedded in an injective $\mathbb{Z}$-module.   4

   (c) Define semi-simple module and prove that every homomorphic image of a semi-simple module is semi-simple.   4

UNIT–V

9. (a) Define Artinian and Noetherian modules and give one example of each. Prove that if a module M is such that it has a submodule N with both N and M / N Noetherian, the M is Noetherian.   7

   (b) Define composition series of a non-zero module and module of finite length. Prove that a module is of finite length if and only if it is both Artinian and Noetherian.   7

10. (a) Define nilradical and Jacobson radical of a ring and also give one example of each. State and prove Nakayama lemma.   8

   (b) Prove that every Artinian ring is Noetherian.   6

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