May 2025
M.Sc.
Fourth Semester
CORE – 11
MATHEMATICS
Course Code: MMAC 4.11
(Mathematical Methods)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Evaluate the integral: 2×3=6

(i) $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

(ii) $\int_0^\infty t e^{-2t} \cos t \, dt$

(b) If $\mathcal{L}\{F(t)\} = f(s)$, then prove that

$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$ where $n = 1, 2, 3, \ldots$. 4

(c) Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2, if |x| \le 1 \\ 0, \quad if |x| > 1 \end{cases}$. 4

2. (a) Evaluate the integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$ using Parseval's identity. 4

(b) Solve the partial differential equation $\frac{\partial U}{\partial t} = 2 \frac{\partial^2 U}{\partial x^2}$, subject to the conditions $U(0,t) = 0$, $U(5,t) = 0$, $U(x,0) = 10 \sin 4\pi x$. 5

(c) Solve the simultaneous ordinary differential equation $\left.\begin{array}{l} Y'+Z''=t \\ Y''-Z=e^{-t} \end{array}\right\}$, using Laplace transform, subject to the conditions $Y(0)=3, Y'(0)=-2, Z(0)=0$. 5

UNIT–II

3. (a) Verify that $\varphi(x)=\frac{x}{(1+x^2)^{5/2}}$ is a solution of the integral equation $\varphi(x)=\frac{3x+2x^3}{3(1+x^2)^2}-\int_0^x \frac{3x+2x^3-t}{(1+x^2)^2}\varphi(x)\,dt$. 4

(b) Form an integral equation corresponding to the initial value problems $\frac{d^3Y}{dX^3}-5XY=0 : Y(0)=\frac{1}{2}, Y'(0)=Y''(0)=1$. 4

(c) Solve the integral equation $\varphi(x)=1+x+\int_0^x (x-t)\varphi(t)\,dt$, using the successive approximation method. 6

4. (a) Obtain the Fredholm integral equation for the differential equation $Y''+XY=1;\ Y(0)=0, Y(1)=1$. 4

(b) Solve the Fredholm's integral equation $\varphi(x)=\cos x+\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\,\varphi(t)\,dt$, using the method of successive substitution. 6

(c) Solve the integral equation $\varphi(x)-\lambda\int_0^{\frac{\pi}{2}} \sin x \cos t\,\varphi(t)\,dt=\sin x$. 4

UNIT–III

5. (a) Using Fredholm's determinant find the resolvent kernel of the kernel $K(x,t)=x^2t-xt^2 : 0\le x\le 1, 0\le t\le 1$, and hence solve the Fredholm integral equation $\varphi(x)=x^2+\lambda\int_0^x K(x,t)\varphi(t)\,dt$. 9

(b) Using Laplace transform, solve the integral equation

$\varphi(x)=\cos x-x-2+\int_0^x(t-x)\varphi(t)\,dt$. 5

6. (a) Find the characteristic number and eigenfunction of the homogenous integral equation $\varphi(x)-\lambda\int_0^\pi K(x,t)\varphi(t)\,dt=0$,

where $K(x,t)=\begin{cases}\cos x\sin t, 0\le x\le t\\ \cos t\sin x, t\le x\le\pi\end{cases}$. 5

(b) Prove that the integral equation $\varphi(x)=\lambda\int_0^1\left(t\sqrt{x}-x\sqrt{t}\right)\varphi(t)\,dt$ does not have real characteristic number and eigenfunction. 4

(c) In a homogeneous Fredholm integral equation with symmetric kernel, prove that every pair of eigenfunction corresponding to different eigenvalue is orthogonal. 5

UNIT–IV

7. (a) Prove that convolution is associative. 4

(b) Calculate the value of $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$, where s is the differential operator. 3

(c) Show that $\dfrac{\{e^t-\sin t-\cos t\}}{\{\sin t\}}=\{2e^t\}$. 3

(d) Show that $l^3\{n^2\cos nt\}=l^2-\left\{\dfrac{1}{n}\sin nt\right\}$, where l is the integral operator. 4

8. (a) Show that $\lim_{a\to\infty}F(a,x)=\lim_{a\to\infty}\dfrac{1}{\pi}\dfrac{a}{x^2+a^2}$ is a Dirac delta function. 4

(b) Show that: 3×2=6

(i) $\delta(x^2-a^2)=\dfrac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right], a\ne 0$

(ii) $f(x)\delta(x-a)=f(a)\delta(x-a)$

(c) Find the Fourier and Laplace transforms of the Dirac delta function. 4

UNIT–V

9. (a) Define a regular Sturm-Liouville problem. Rewrite the Bessel's equation $x^2y''+xy'+\left(\lambda x^2-n^2\right)y=0$ in Sturm-Liouville form.

1+2=3

(b) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem 5

$$y''+\lambda y=0, 0<x<L$$

$$y(0)=0, y(L)=0$$

(c) Express the function $f(x) = 1$ as the eigenfunction series of the Sturm-Liouville problem 6

$$y''+\lambda y=0, 0<x<L$$

$$y(0)=0, hy(L)+y'(L)=0, h>0$$

10. (a) Find the Green's function for the boundary-value problem 7

$$y''-k^2y=0, k\neq 0$$

$$y(0)=y(1)=0$$

(b) Reduce the boundary value problem to an integral equation using Green's function 7

$$y''+\lambda y=x^2, 0<x<\frac{\pi}{2}$$

$$y(0)=y\left(\frac{\pi}{2}\right)=0$$