May 2025
M.Sc.
Second Semester
CORE – 08
MATHEMATICS
Course Code: MMAC 2.41
(Complex Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Define a power series. Show that a power series represents a continuous function at each point inside the circle of convergence. 1+4=5

(b) If a power series converges to $f(z)$ at all points inside the circle $|z-z_0|=R$, then prove that it is the Taylor series expansion of *f* in powers of $z-z_0$. 5

(c) Obtain an expansion for $f(z)=\frac{1}{z}$ in powers of $z-1$ in the region $(|z-1|<1)$, and hence prove that

$\frac{1}{z^2}=\sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n$, in the same region. 4

2. (a) Show that the sum $S(z)$ of the power series $\sum_{n=0}^{\infty}a_n(z-z_0)^2$ is analytic at each point *z* interior to the circle of convergence of that series. 5

(b) Define the term circle of convergence. Obtain the radii of convergence of the power series: 1+2+2=5

(i) $\sum_{n=0}^{\infty}\frac{z^2}{n!}$

(ii) $\sum_{n=0}^{\infty} \frac{z^2}{2^n + 1}$

(c) When do you say that a series of complex numbers is absolutely convergent? Does the absolute convergence of a series of complex number imply the convergence of the series? Justify. 1+3=4

UNIT–II

3. (a) State and prove the Cauchy's residue theorem. 1+4=5

(b) Write the two Laurent series in powers of z that represents the function $f(z) = \frac{1}{z(1+z^2)}$. 5

(c) Find the residue at $z = i$ for 2×2=4

(i) $\frac{z^3 + 2z}{(z-i)^3}$

(ii) $\frac{(\log z)^3}{z^2 + 1}$

4. (a) Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then show that z_0 is the simple pole of the quotient $p(z)/q(z)$ and $\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'R(z_0)}$. 5

(b) Derive the Maclaurin series for the function $f(z) = \cos z$ by

(i) using the definition $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ 2½×2=5

(ii) showing that $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$

(c) Find the value of the integral $\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)} dz$ taken counterclockwise around the circle $|z-2| = 2$. 4

UNIT–III

5. (a) Use residues to evaluate the improper integral $\int_0^\infty \frac{x^2}{x^6+1}dx$. 7

(b) Derive the integration formula $\int_0^\infty \frac{\ln x}{(x^2+4)^2}dx = \frac{\pi}{32}(\ln 2 - 1)$. 7

6. (a) Find the Cauchy principal value of the integral
$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}dx$. 7

(b) Use residues to evaluate the definite integral $\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2}$. 7

UNIT–IV

7. (a) Define winding number. Let Γ be the image of C, a positively oriented unit circle, under the transformation $w = \frac{1}{z^2}$. Show that Γ winds around the origin twice in the clockwise direction. 2+3=5

(b) Find the function $f(t)$ that corresponds to

$$F(s) = \frac{s}{\left(s^2+a^2\right)^2}\left(a>0\right).$$

5

(c) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. 4

8. (a) Find $f(t)$ when $F(s) = \frac{\tanh s}{s^2}$. 6

(b) Find the winding number of the curve $f(\theta) = 2e^{(2i\theta)}$ for $0 \le \theta \le \pi$ around the point $|z| = 0$. 4

(c) Determine the number of roots of the equation $z^7 - 4z^3 + z - 1 = 0$ inside the circle $|z| = 1$. 4

UNIT–V

9. (a) Show that the transformation $w = iz + i$ maps the half plane $x > 0$ onto the half plane $v > 1$. 5

(b) Determine the angle of rotation at the point $z = 2 + i$ when the transformation is $w = z^2$, illustrating it for some particular curve. Also, find the scale factor of the transformation at that point. 3+2=5

(c) Define the terms conformal mapping, isogonal mapping, and angle of rotation. 4

10. (a) Show that when $c_1 < 0$, the image of the half plane $x < c_1$ under the transformation $w = \frac{1}{z}$ is interior of a circle. What is the image when $c_1 = 0$? 5

(b) Show that $f(z) = z^2$ is conformal at the point $z = 1 + i$, considering the half lines $y = x\,(x \geq 0)$ and $x = 1\,(x \geq 0)$. 5

(c) Find a linear fractional transformation that maps the points $z_1 = 1,\ z_2 = 0,\ z_3 = -1$ onto the points $w_1 = i,\ w_2 = \infty,\ w_3 = 1$. 4