May 2025
M.Sc.
Second Semester
CORE – 06
MATHEMATICS
Course Code: MMAC 2.21
(General Topology)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Let (X, τ) be a topological space, and let B be a basis that generates τ. Prove that every open set in X can be expressed as a union of elements of B. 5

   (b) Let X be an infinite set and consider the following two topologies:

   τ_{cf} (the cofinite topology), and

   τ_d (the discrete topology).

   (i) Prove that τ_d is strictly finer than τ_{cf}. 3

   (ii) Show that for every finite subset $F \subseteq X$, the two induced subspace topologies on F coincide and are both discrete. 3

   (iii) Give an example of a set that is open in τ_d but not in τ_{cf}. 3

2. (a) Prove that the intersection of all topologies containing a given subbasis $\mathcal{S}$ is exactly the topology generated by $\mathcal{S}$. 5

   (b) In a topological space (X, τ), define the exterior of a set $A \subseteq X$. Prove that A is closed if and only if the exterior of A is empty. 5

   (c) Provide an example of a topology on $\mathbb{R}$ (other than the standard or lower limit topology) that cannot be induced by any metric. Explain why it is non-metrizable. 4

UNIT–II

3. (a) State and prove the pasting lemma for continuous functions and give an example illustrating its use. 5+2=7

(b) Let X, Y, Z be topological spaces and let $f : Z \to X$, $g : Z \to Y$ be continuous. Show that the induced map
$F : Z \to X \times Y$, $F(z) = (f(z), g(z))$, is continuous. 3

(c) Let $\sim$ be an equivalence relation on a space X, and let
$q : X \to X/\sim$ be the quotient map. Show that if each equivalence class is closed in X, then q is a closed map. 4

4. (a) Let (X, τ) be a topological space and let $f : X \to Y$ be a function into a topological space (Y, σ).

(i) Define what it means for f to be sequentially continuous at $x \in X$. 2

(ii) Prove that if f is continuous at x, then f is sequentially continuous at x. Is the converse true? Justify with an example. 5

(b) Suppose $f : X \to Y$ is a closed, continuous, and surjective map. Prove that Y, endowed with the quotient topology induced by f, is homeomorphic to Y with its original topology. 7

UNIT–III

5. (a) Prove that a space X is disconnected if and only if there exists a non-constant continuous map $X \to D$, where D is a discrete two-point space. 7

(b) Give an example of a topological space that is connected but not path-connected and justify why it has these properties. 7

6. (a) Define locally connected space. Prove that every open subspace of $\mathbb{R}^n$ (in the usual topology) is locally connected. 4

(b) Prove that the continuous image of a path-connected space is path-connected. 3

(c) Let X be covered by finitely many connected subspaces whose pairwise intersections are non-empty. Show that X itself is connected. 7

UNIT–IV

7. (a) Prove that any continuous bijection from a compact space onto a Hausdorff space is a homeomorphism. 7

   (b) Let X be compact and Y be Hausdorff. Show that if $f, g : X \to Y$ are continuous maps which agree on a dense subset of X, then $f = g$ on all of X. 7

8. (a) If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, prove that $f(K)$ is compact in Y. 4

   (b) Prove that a compact, connected subset of $\mathbb{R}$ is an interval. 4

   (c) Let X be a locally compact Hausdorff space and let $x \in X$. Prove that there exists some neighbourhood U of x whose closure $\overline{U}$ is compact in X. 6

UNIT–V

9. (a) Show that every normal (T_4) space is regular (T_3). 7

   (b) Define a Lindelöf space. Prove that every second-countable space is Lindelöf. Is every Lindelöf space second-countable? Justify with an example or counterexample. 7

10. (a) Prove that every metric space is normal. 7

    (b) Distinguish among first countable, second countable, and separable topological spaces. Give one example of each concept, with justification. 7
