2024
M.Sc.
Fourth Semester
CORE – 12
MATHEMATICS
Course Code: MMAC 4.21
(Rings & Modules)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Prove that every field is an integral domain. Is the converse true? Justify. 4
   (b) If R and S are any two rings, prove that their Cartesian product $R\times S$ is also a ring under coordinate wise addition and multiplication. 6
   (c) Show that any ring can be embedded into a ring with unity. 4

2. (a) If G is an additive abelian group, prove that the set of all group endomorphisms of G is a ring under pointwise addition and composition of maps. 5
   (b) Define local ring, provide some examples and prove that the characteristic of a local ring is either 0 or a power of a prime. 5
   (c) If I is a left ideal and J is a right ideal of a ring R, prove that IJ is a 2-sided ideal of R. 4

UNIT–II

3. (a) Prove that unitary modules over $\mathbb{Z}$ are simply abelian groups. 5
   (b) If N is a submodule of an R-module M, show that the quotient group M/N has a natural structure of an R-module. 5
   (c) If R is any ring, show that R is naturally a left R-module and also a right R-module with usual multiplication in R as the scalar multiplication. 4

4. (a) If M is an abelian group, R is a ring and $End_Z(M)$ is the ring of all additive endomorphisms of M, prove that M is a left R-module if and only if there exists a ring homomorphism $\theta : R \to End_Z(M)$. 5

(b) State and prove the epimorphism theorem for modules. 5

(c) Show that a submodule P of an R-module M is a direct summand of M if and only if there is a projection p of M onto P, i.e., $p(M) = P$. 4

UNIT–III

5. (a) If M and N are R-modules, show that their Cartesian product $M \times N$ can also be made into an R-module in a natural way. 5

(b) Show that the cardinalities of any two bases of a free module F over a commutative ring R with 1 are equal. 4

(c) Prove that every free module is a projective module and justify that the converse is not true by an example. 5

6. (a) If R is an integral domain, define torsion submodule of an R-module M and show that the set of torsion elements of M forms a submodule of M. 4

(b) Prove that an R-module P is projective if and only if every exact sequence of the form $0 \to M' \to M \to P \to 0$ splits. 6

(c) Prove that any unitary module over a ring with unity is a quotient of a free module. 4

UNIT–IV

7. (a) Define simple module and also give an example. State and prove Schur's lemma. 4

(b) If $\{M_i\}_{i \in I}$ is a family of R-modules, prove that $\prod_{i \in I} M_i$ is injective if and only if each M_i is injective. 6

(c) Show that every injective module is divisible. 4

8. (a) Show that an R-module E is injective if the functor $Hom_R(\ , E)$ is exact. 6

(b) Prove that every $\mathbb{Z}$-module can be embedded in an injective $\mathbb{Z}$-module. 4

(c) Define semi-simple module and prove that a direct sum of semi-simple modules is semi-simple. 4

UNIT–V

9. (a) Prove that if a module M is such that it has a submodule N with both N and M/N Artinian, then M is Artinian. 4

(b) If J is the Jacobson radical of A then show that $x \in J$ if and only if $1 - xy$ is a unit in A for all $y \in A$. 6

(c) Show that a commutative Artinian ring is Noetherian. 4

10. (a) Define nil radical $N(R)$ and Jacobson radical $J(R)$ of a ring R and show that $N(R) \subseteq J(R)$. 3

(b) State and prove Jordan-Hölder theorem. 7

(c) Show that a semi-simple ring is of finite length. 4