2024
M.Sc.
Second Semester
CORE – 08
MATHEMATICS
Course Code: MMAC 2.41
(Complex Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Let $z_n = x_n + iy_n \left(n = 1, 2, 3, \ldots\right)$ and $S = X + iY$. Prove that

$\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. 5

(b) Let z_1 be a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$. Prove that the series must be uniformly convergent in the closed disk $|z - z_0| \le R_1$, where $R_1 = |z_1 - z_0|$. 5

(c) Obtain the Maclaurin series representation for $\frac{1}{1-z}\left(|z| < 1\right)$, and hence, differentiating it, obtain the expansion of $\frac{1}{\left(1-z\right)^2}$. 4

2. (a) If a power series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$ converges when $z = z_1 \left(z_1 \ne z_0\right)$, then prove that it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$. 5

(b) Prove that the power series $\sum_{n=0}^{\infty} a_n (z = z_0)^n$ can be differentiated term by term. 5

(c) Show that if $\lim_{x\to\infty} z_n = z$, then $\lim_{x\to\infty} |z_n| = |z|$. 4

UNIT–II

3. (a) Expand $f(z) = \frac{z+3}{(z^2 - z - 2)z}$ in powers of z 5

   (i) within the unit circle about the origin

   (ii) within the annulus region between the concentric circles about the origin having radii 1 and 2 respectively

   (iii) the exterior of the circle with center as origin and radius 2

   (b) Prove that a function f that is analytic at a point z_0 has a zero of order m there if and only if there is a function g, which is analytic and non-zero at z_0, such that $f(z) = (z - z_o)^m g(z)$. 5

   (c) Find the residue of $f(z) = \frac{z^3}{z^2 - 1}$ at $z = \infty$. 4

4. (a) Prove that an isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

   $f(z) = \frac{\phi(z)}{(z - z_o)^m}$, where $\phi(z)$ is analytic and non-zero at z_0. 5

   (b) Discuss the different types of isolated singularities,with suitable examples. 5

   (c) Find the residues at the poles of the function

   $$f(z) = \frac{z^4}{(a^2 + z^2)^4}$$ 4

UNIT–III

5. (a) Evaluate $\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$. 7

(b) Use the method of contour integration to show that

$$\int_0^\infty \frac{x^6}{\left(a^4+x^4\right)}dx = \frac{3\pi\sqrt{2}}{16a}, a>0.$$ 7

6. (a) Evaluate $\int_0^\infty \frac{x^{-a}}{x+1}dx, (0<a<1)$. 7

(b) Prove that $\int_0^\infty \frac{\sin x}{x\left(x^2+a^2\right)}dx = \frac{\pi}{2a^2}\left(1-e^{-a}\right), (a>0)$. 7

UNIT–IV

7. (a) If $f(z)$ is meromorphic inside a closed contour C and has no zero on C, then show that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = Z-P,$ where Z is the number of zeros and P is the number of poles, counting multiplicities. 5

(b) Determine the number of zeros, using the argument principle, of $z^5+2z^3-z^2+2z+5$ that lies in the first quadrant. 5

(c) Use Rouché's theorem to show that the equation $z^5+15z+1=0$ has one root in the disk $|z|<\frac{3}{2}$ and four roots in the annulus $\frac{3}{2}\le|z|<2$. 4

8. (a) By using Rouché's theorem, prove that every polynomial $P(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n=0, \left(n\ge 1, a_n\ne 0\right)$ has exactly n roots. 5

(b) State and prove the Rouché's theorem. 5

(c) Determine the number of zeros, counting multiplicities, of the polynomial $2z^5-6z^2+z+1$ in the annulus $1\le|z|<2$. 4

UNIT–V

9. (a) Show that the set of all bilinear transformation formsa non-Abelian group under the composition of transformation. 5

(b) Show that the transformation $w = \frac{i-z}{i+z}$ maps the half plane $\operatorname{Im} z > 0$ onto the disk $|w| < 1$ and the boundary of the half plane onto the boundary of the disk. 5

(c) Find the image of the quadrant $x > 1, y > 0$ under the transformation $w = \frac{1}{z}$. 4

10. (a) Discuss the conformality of the function $f(z) = z^2$ at the point of intersection of the half lines $y = x, (x \geq 0)$ and $x = 1 (y \geq 0)$. Determine the scale factor and the angle of rotation at that point. 5

(b) Show that the transformation $w = \frac{1}{z}$ transforms circles and straight line to circles and straight lines. 5

(c) Find the linear fractional transformation that maps $z_1 = \infty, z_2 = i, z_3 = 0$ onto the points $w_1 = 0, w_2 = i, w_3 = \infty$. 4