2024

M.Sc.

Second Semester

CORE – 06

MATHEMATICS

Course Code: MMAC 2.21

(General Topology)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) For a given topological space, define interior, closure, and boundary of a set. Use the set $A=\left\{\frac{1}{n}: n\in \mathbb{N}\right\}$ in $\mathbb{R}$ with the standard topology to illustrate these concepts. 2+2+2=6

(b) Explain the difference between a basis and a sub-basis for a topology. Illustrate with an example how a topology can be generated from a sub-basis. 2+2=4

(c) Defne what it means for one topology to be finer than another. Provide an example of two different topologies on the same set where one is finer than the other. 2+2=4

2. (a) Defne a metric topology and illustrate with an example of a metric that induces the standard topology on $\mathbb{R}$. Explain how the concepts of open sets in a metric space relate to those in a general topological space. 3+3=6

(b) Let (X,τ)) be a topological space. Let $A\subseteq X$ and A' is the set of all limit points of *A*. Then, prove that $\overline{A}=A\cup A'$. 4

(c) If $\mathcal{B}$ is a basis for a topology τ on *X*, then prove that τ is unique. 4

UNIT–II

3. (a) Let (X, τ) and (X, τ') be two topological spaces and $f : X \to Y$ be a function, then prove that f is continuous if and only if for every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$. 7

   (b) Let (X, τ) be a topological space; let A be a set; let $p : X \to A$ be a surjective map. Show that the quotient topology on A induced by p is the largest topology relative to which p is continuous. 7

4. (a) Show that the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$. 4

   (b) Let $f : X \to Y$ and $g : Y \to Z$ be functions between topological spaces. Assume f is continuous at a point $x \in X$ and g is continuous at $f(x)$. Prove that the composition $g \circ f : X \to Z$ is continuous at x. 4

   (c) Define what it means for a function between two topological spaces to be open and to be closed. Provide an example of a function that is both open and closed. 2+4=6

UNIT–III

5. (a) Show that the image of a connected topological space under a continuous map is connected. Give an example. 5+2=7

   (b) If $\{A_\alpha\}$ is a collection of path connected subsets of X and if $\cap A_\alpha \neq \emptyset$, is $\cup A_\alpha$ necessarily path connected? 7

6. (a) Prove that the topological space $(\mathbb{R}, \tau_{std})$ is connected. 7

   (b) What are the components and path components of $(\mathbb{R}, \tau_l)$? Explain. 3½+3½=7

UNIT–IV

7. (a) Let (X, τ) and (Y, τ') be compact topological spaces. Prove that the product space $X \times Y$, equipped with the product topology, is compact. 7

(b) Let X be a compact Hausdorff space. Show that if $\{A_n\}$ is a countable collection of closed sets in X, each of which has an empty interior in X, then there is a point of X which is not in any set A_n. 7

8\. (a) Prove that every closed subset of a compact topological space is compact. Is a compact set in a topological space necessarily closed? 5+2=7

(b) Suppose $\mathcal{F}$ is a family of closed subsets of a compact space X such that $\mathcal{F}$ has the finite intersection property. Prove that the intersection of all sets in $\mathcal{F}$ is non-empty. Give an example of a compact space and a family of closed sets with the finite intersection property to illustrate your proof. 7

UNIT–V

9\. (a) Prove that a topological space (X, τ) is a T_1 space if and only if every singleton set $\{x\}$ is closed. Illustrate with an example of a topological space that is T_1 but not T_2, and discuss how the property differ. 4+2=6

(b) Justify the following statements:

(i) If X is a second countable space, then X is a Lindelöf space

(ii) Every compact space is Lindelöf, but not all Lindelöf spaces are compact. 4+4=8

10\. (a) Prove that a topological space (X, τ) is classified as T_3 if, for every $x \in X$ and any open set U that includes x, there is another open set V in which x is contained, and the closure of V, denoted as $\overline{V}$, is completely contained within U. 7

(b) Why all normal spaces are regular? And why the converse is not true? 3½+3½=7