2024

B.A./B.Sc.

Sixth Semester

DISCIPLINE SPECIFIC ELECTIVE – 4

MATHEMATICS

Course Code: MAD 6.21

(Differential Geometry)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Find the equation of the osculating plane at a general point on the cupid $\overline{r} = (t, t^2, t^3)$ and show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these three points. 5

   (b) Show that the necessary and sufficient condition for the curve to be a straight line is that the curvature $\kappa = 0$ at all points of the curve. 4

   (c) For the curve $x = a(3t - t^3)$, $y = 3at^2$, $z = a(3t + t^3)$. Show that

   $$\kappa = \tau = \frac{1}{3a(1+t^2)^2}.$$ 5

2. (a) State and prove existence theorem. 5

   (b) Show that the radius of the spherical curvature of a circular helix is $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$ is equal to the radius of the circular curvature. 4

   (c) Find the radius and the centre of the sphere of curvature. 5

UNIT–II

3. (a) Prove that at points common to the surface $a(yz + zx + xy) = xyz$ and a sphere whose center is the origin, the tangent plane to the surface makes intercepts on the axis whose sum is constant. 5

(b) Show that the metric or the first fundamental form is a positive definite quadratic form in du, dv. 3

(c) State and prove Rodrigue's formula. 6

4. (a) Find the equations giving the principal curvatures. 6

(b) Prove that the normals to any surface at consecutive points of one of its line of curvature intersets. Conversely if the normals at two consecutive points of a curve drawn on a surface intersect then the curve is a line of curvature. 5

(c) Find the angle between 2 directions on the surface at the point P having the direction coefficients (l, m) and (l', m') 3

UNIT–III

5. (a) State and prove Clairaut's theorem. 5

(b) Show that on a right circular cone $x = u, y = u, z = u \cot \alpha$ the geodesics are given by $u = h \sec(v \sin \alpha + \beta)$ where h, α and β are constants. 6

(c) Prove that if the parametric curves on a surface are orthogonal the curves v = constant will be geodesics provided E is a function of u only and the curve u = constant will be geodesic if G is a function of v only. 3

6. (a) State and prove Gauss Bonnet theorem. 5

(b) Prove that the curves of the family $\frac{v^3}{u^2}$ = constant are geodesic on the surface with metric $v^2du^2 - 2uvdudv + 2u^2du^2$ $(u, v > 0)$. 4

(c) If a mapping is geodesic and conformed, show that it is an isometry or else a similarity mapping. 5

UNIT–IV

7. (a) Prove that the equations of transformation of a mixed tensor possess the group property. 5

(b) Show that any linear combination of the tensor of the type (r, s) is a tensor of the type (r, s). 4

(c) Show that a second rank covariant tensor of the type (2,0) is expressible as a sum of two tensors, one of which is symmetric and the other is anti-symmetric. 5

8. (a) State and prove Quotient Law. 5

(b) The tensor product of the tensors of the type (r, s) and (r', s') is a tensor of the type $(r+r', s+s')$. 5

(c) If the tensors a_{ij} and g_{ij} are symmetric and u^i, v^i are contravariant vectors satisfying the equations

$(a_{ij} - kg_{ij})u^i = 0$

$(a_{ij} - k'g_{ij})v^i = 0$ with $k \neq k'$

prove that $g_{ij}u^i v^j = 0$, $a_{ij}u^i v^j = 0$ 4

UNIT–V

9. (a) Obtain the tensor laws of transformation of Christoffel symbols. 6

(b) Find the covariant derivative of a covariant vector. 4

(c) Prove that the necessary and sufficient condition that all the Christoffel symbols vanish at a point is that g_{ij} are constant. 4

10. (a) Prove that the necessary and sufficient condition that the curl of a vector field vanishes is that the vector field be gradient. 5

(b) If at a specified point, the derivatives of g_{ij} with respect to x^k are all zero, prove that the components of covariant derivatives at that point are the same as ordinary derivatives. 4

(c) If ϕ is a scalar function of co-ordinates x^i, then

$$\text{div grad } \phi = \nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \frac{\partial\phi}{\partial x^l}\begin{Bmatrix} l \\ ij \end{Bmatrix}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij}\phi_{,j})$$

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