2024

B.A./B.Sc.

Sixth Semester

CORE – 14

MATHEMATICS

Course Code: MAC 6.21

(Ring Theory & Linear Algebra - II)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If D is an integral domain and $f(x),\ g(x) \in D[x]$, prove that $\deg(f(x).g(x)) = \deg f(x) + \deg g(x)$.
Show, by example, that for commutative ring R it is possible that $\deg(f(x).g(x)) < \deg f(x) + \deg g(x)$, where $f(x)$ and $g(x)$ are non-zero elements in $R[x]$. 4

(b) For every prime p, show that $x^{p-1} - 1 = (x-1)(x-2)...(x-(p-1))$ in $\mathbb{Z}_p[x]$. 5

(c) If F is a field, I a non-zero ideal in $F[x]$ and $g(x)$ an element of $F[x]$, show that $I = \langle g(x) \rangle$ if and only if $g(x)$ is a non-zero polynomial of minimum degree in I. 5

2. (a) Prove or disprove whether reducibility over $\mathbb{Q}$ implies reducibility over $\mathbb{Z}$. 4

(b) State and prove Eisenstein's criterion for irreducibility of polynomials. 6

(c) Define zero of a polynomial and show that $x^2 + 3x + 2$ has four zeros in $\mathbb{Z}_6$. 2

(d) Show by means of an example that Mod p irreducibility test may fail for some prime p and work for others. 2

UNIT–II

3. (a) Prove that the ring of integers is a Euclidean domain. 4

(b) Show that any two non-zero elements in a unique factorization domain (UFD) have a g.c.d. 6

(c) If D is a Euclidean domain with measure d, prove that u is a unit in D if and only if $d(u) = d(1)$. 4

4. (a) Prove that every PID is a UFD. Is the converse true? Justify. 8

(b) Define prime and irreducible elements and show that in a UFD, an element is prime if and only if it is irreducible. 6

UNIT–III

5. (a) If $B = \{(-1,1,1),(1,-1,1),(1,1,-1)\}$ is a basis of $\mathbb{R}^3$, find the dual basis of B. 5

(b) If W is a subspace of a finite dimensional vector space V over a field F, prove that $\dim W + \dim W^0 = \dim V$, where W^0 is the annihilator of W. 5

(c) If T is a linear operator on a finite dimensional vector space V over a field F and $\alpha \in F$ is an eigen value of T, show that for any polynomial $p(x) \in F(x)$, $p(\alpha)$ is an eigen value of $p(T)$. 4

6. (a) Define invariant subspace and if T is a linear operator on a finite dimensional vector space V, show that kernel of T and image of T are invariant under T. 4

(b) If T is a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$, prove that T is diagonalizable by exhibiting a basis of $\mathbb{R}^3$, each of which is an eigen vector of T. 6

(c) If T is a linear operator on a finite dimensional vector space V, show that the roots of the minimal polynomial and the characteristic polynomial of T are same, except for their multiplicities. 4

UNIT–IV

7. (a) State and prove Cauchy-Schwarz inequality for inner product space. 4

(b) Find an orthonormal basis of the vector space V of all real polynomials of degree not greater than 2 in which the inner product is defined as $\langle \theta(x), \psi(x) \rangle = \int_0^1 \theta(x)\psi(x)dx$. 6

(c) Define orthogonal complement and show that if W is a non-empty subset of an inner product space V over a field F, then the orthogonal complement of W, denoted by $W^{\perp}$ is a subspace of V. 4

8. (a) State and prove Bessel's inequality. 6

(b) Is an orthogonal set of non-zero vectors of an inner product space linearly independent? Justify. 4

(c) Show that an inner product over a vector space V over $\mathbb{R}$ is a bilinear form. 4

UNIT–V

9. (a) If V is a finite dimensional inner product space and T is a linear operator on V, prove that there exists a unique function $T^*: V \to V$ such that $\langle T(\mathrm{x}), \mathrm{y} \rangle = \langle x, T^*(y) \rangle \forall x,\ y \in V$. Also, show that T^* is linear. 5

(b) If T is a self-adjoint operator on an inner product space V, show that every eigen value of T is real and that eigen vectors of T corresponding to distinct eigen values are orthogonal. 5

(c) If T is a normal operator on an inner product space V over a field F, show that $T - \alpha I$ is normal for every $\alpha \in F$. 4

10. (a) If T is a linear operator on a finite dimensional complex inner product space V, prove that T is normal if and only if there exists an orthonormal basis for V consisting eigen vectors of T. 5

(b) If V is an inner product space and T is a linear operator on V, show that T is an orthogonal projection if and only if T has an adjoint T^* and $T^2 = T = T^*$. 6

(c) Show that the product of two self-adjoint operators on an inner product space is self-adjoint if and only if the two operators commute. 3