2024
B.A./B.Sc.
Sixth Semester
CORE – 13
MATHEMATICS
Course Code: MAC 6.11
(Metric Spaces & Complex Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If $X = \mathbb{R}^n$ and define $d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, p \geq 1 \forall x, y \in X$.
Show that (X, d_p) is a metric space. 7
(b) Let X be a metric space. Then show that
(i) any union of open sets in X is open
(ii) any finite intersection of open sets in X is open 7

2. (a) Show that a subset E of a metric space (X, d) is closed if and only if E contains all its limit points. 6
(b) Define Cauchy sequence. Show that if (X, d) be a metric space, then 4×2=8
(i) any convergent sequence in (X, d) is Cauchy. Show also that converse is not true.
(ii) any Cauchy sequence (x_n) in X is bounded, that is all $x_n \in \beta(x, r)$ for some $x \in X$ and $r > 0$.

UNIT–II

3. (a) Let (X, d) be a metric space. Assume that D is a dense subset of X. Let Y be a complete metric space. Let $f : (D, d) \to (Y, d)$ be a

uniformly continuous function. Then show that there exist a uniformly continuous function $g : X \rightarrow Y$ such that $g(x) = f(x) \forall x \in D$. 7

(b) Let X be a metric space, let Y be a complete metric space and let A be a dense subspace of X. If f is a uniformly continuous mapping of A into Y, then f can be extended uniquely to a uniformly continuous mapping of g of X into Y. 7

4. (a) Let X and Y be connected metric spaces. Then, the product space $X \times Y$ is connected. 6

(b) Show that 4×2=8

(i) If X be a metric space, and A and B be two connected subsets of X such that $A \cap B \neq \phi$. Then $A \cup B$ is connected.

(ii) If A be a connected subset of a metric space X and $A \subset B \subset \overline{A}$ then B is connected.

UNIT–III

5. (a) If $f(z)$ and $g(z)$ are continuous at $z = z_0$, prove that $3f(z) - 4ig(z)$ is also continuous at $z = z_0$. 4

(b) If z_1, z_2 are any complex numbers, then

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$ and deduce that

$\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$. 6

(c) Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at origin. 4

6. (a) State and prove the sufficient conditions for differentiability. 5

(b) Show that the function $f(x) = \sqrt{|xy|}$ is not analytic at origin although the Cauchy Riemann equation are satisfied. 4

(c) If $f(z)=\dfrac{x^3 y(y-ix)}{x^6+y^2}, z\neq 0$ and $f(0)=0$. Show that $\dfrac{f(z)-f(0)}{z}\to 0$ as $z\to 0$ along any radius vector but not as $z\to 0$ in any manner. 5

UNIT–IV

7. (a) Evaluate $\displaystyle\int_{(0,3)}^{(2,4)}\left(2y+x^2\right)dx+\left(3x-y\right)dy$ along: 7

(i) the parabola $x=2t, y=t^3+3$

(ii) the straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$.

(b) Verify Cauchy's theorem for the function $3z^2+iz-4$ if C is the square with vertices at $1\pm i, -1\pm i$. 7

8. (a) If D be a simply connected region and let $f(z)$ be a single valued continuously differentiable function on D i.e. $f'(z)$ exist and is continuous at each points of D. Then show that $\displaystyle\int_c f(z)dz=0$. 6

(b) Evaluate $\displaystyle\int_c \frac{e^{2z}}{(z+1)^4}dz$ where the path of integration C is $|z|=3$. 4

(c) Evaluate $\displaystyle\frac{1}{2\pi i}\int_c \frac{e^3}{z-2}dz$ if C is the circle 4

(i) $|z|=3$ (ii) $|z|=1$

UNIT–V

9. (a) Expand $\dfrac{1}{z^2-3z+2}$ for 5

(i) $1<|z|<2$ (ii) $|z|>2$

(b) Show that $\log(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots$, when $|z-1| < 1$ 4

(c) Obtain the Taylor's and Laurent series which represents the function $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ in the region

(i) $|z| < 2$ (ii) $2 < |z| > 3$ 5

10. (a) Define uniform convergence of series. Show that the following series is absolutely and uniformly convergent for all values of z, real or complex $1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$. 7

(b) If $f(z)$ is analytic inside a circle *C* with centre at *a*, then for all *z* inside *C*. Show that

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

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