2024
B.A./B.Sc.
Fourth Semester
CORE – 10
MATHEMATICS
Course Code: MAC 4.31
(Ring Theory & Linear Algebra - I)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Define the center of a ring. Prove that the center of a ring is a subring. 1+4=5
   (b) Let R be a commutative ring with characteristic p. Show that
   $(x+y)^p = x^p + y^p, \forall x, y \in R$ 5
   (c) Show that the ring of integers modulo p, $\mathbb{Z}_p$, is a field. 4

2. (a) Let a and b belong to a ring R and let m and n be integers. Prove that $m\cdot(ab) = (m\cdot a)b = a(m\cdot b)$ and $(m\cdot a)(n\cdot b) = (mn)\cdot(ab)$. 5
   (b) Let R be a ring with unity 1. Show that if 1 has infinite order under addition, then the characteristic of R is 0 and if 1 has order n under addition, then the characteristic of R is n. 5
   (c) Define a nilpotent element. If R is a ring with unity 1 and a is a nilpotent element of R, then prove that $1-a$ is a unit. 4

UNIT–II

3. (a) If A is a subring of a ring R, then prove that the set of cosets $\{r + A \mid r \in R\}$ is a ring under operations
   $(s+A)+(t+A) = s+t+A$ and $(s+A)(t+A) = st+A$ if and only if A is an ideal of R. 5

(b) If R is a finite commutative ring with unity, prove that every prime ideal of R is a maximal ideal of R. Comment on the converse too. 5

(c) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z} \right\}$ and

$S = \left\{ \begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \mid r, s, t \in \mathbb{Z}, s \text{ is even} \right\}$. If S is an ideal of R, what can

you say about r and s? 4

4. (a) If A and B are ideals of a ring, show that the product of A and B is also an ideal. 5

(b) Show that $\mathbb{R}[x] / \langle x^2 + 1 \rangle$ is a field. 5

(c) Let R be a commutative ring and let A be any subset of R. Show that the annihilator of A is an ideal. 4

UNIT–III

5. (a) Prove that every ideal of a ring R is the kernel of a ring homomorphism of R. 5

(b) Let ϕ be a ring homomorphism from a ring R to ring S. Let A be a subring of R and B be an ideal of S. Show that $\phi(A)$ is a subring of S and $\phi^{-1}(B)$ is an ideal of R. 5

(c) Determine all ring homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_6$. 4

6. (a) Let R be a ring and $A \subset B$ be ideals of R. Show that $\dfrac{R/A}{B/A} \cong \dfrac{R}{B}$. 5

(b) Let n be an integer whose decimal representation is given as $a_k a_{k-1} \ldots a_1 a_0$. Prove that n is divisible by 3 if and only if $a_k + a_{k-1} + \ldots + a_1 + a_0$ is divisible by 3. 5

(c) Let R be a commutative ring of prime characteristic p. Show that the Frobenius map $f : R \to R$ defined by $f(x) = x^p$ is a ring homomorphism. 4

UNIT–IV

7. (a) Let S be a linearly independent subset of a vector space V and let x be an element of V that is not in S. Show that $S \cup \{x\}$ is linearly dependent if and only if $x \in$ Span(S). 5

(b) Determine if the following subsets of the vector space $\mathbb{R}^3$ for subspaces of $\mathbb{R}^3$ 2½×2=5

(i) $V = \{(a, 2a, 3a) | a \in \mathbb{R}\}$

(ii) $W = \{(a, a^2, b) | a, b \in \mathbb{R}\}$

(c) Find a number t such that (3, 1, 4), (2, –3, 5), (5, 9, t) is not linearly independent in $\mathbb{R}^3$. 4

8. (a) Suppose x_1, x_2, x_3, x_4 are linearly independent in V. Prove that the list $x_1 - x_2, x_2 - x_3, x_3 - x_4, x_4$ is also linearly independent. 5

(b) Let V be the vector space of all 2×2 matrices over the field F. Let W be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$. Prove that W is a subspace of V. 5

(c) If V is a finitely generated vector space, then prove that every basis of V has the same number of elements. 4

UNIT–V

9. (a) State and prove the second isomorphism theorem of linear algebra. 5

(b) Let T be a linear operator on $\mathbb{R}^2$ defined by $T(a, b) = (3a - b, a + 3b)$ where $\beta = \{(1, 1), (1, -1)\}$ and $\beta' = \{(2, 4), (3, 1)\}$ are the ordered basis of $\mathbb{R}^2$.

Find $[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$ where Q is the change of coordinate matrix from β' to β coordinates. 5

(c) Let T be a linear transformation from V to W. Prove that T is injective if and only if $N(T) = \{0\}$. 4

10. (a) If $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $B = \{e_1, e_2\}$ and $C = \{e_1 + e_2, e_1 - e_2\}$ be two ordered bases of $\mathbb{R}^2$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(x, y) = (x + y, x - 2y)$: 2+3=5

(i) Determine the matrix of T with respect to the basis B.

(ii) Determine the matrix of T with respect to the basis C.

(b) Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$. Show that T is linear if and only if $b = c = 0$. 5

(c) Let V and W be vector spaces and $T : V \to W$ be linear. Prove that $N(T)$ and $R(T)$ are subspaces of V and W, respectively. 4
