2024

B.A./B.Sc.

Fourth Semester

CORE – 9

MATHEMATICS

Course Code: MAC 4.21

(Riemann Integration & Series of Functions)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Define upper and lower Darboux sums. Given the function $f(x) = 2x + 1$ on the interval [1, 3], calculate the upper and lower Darboux sums using a partition of the interval into two equal parts. Discuss how these sums provide bounds for the integral of *f*. 2+3+2=7

   (b) A bounded function *f* on the interval $[a, b]$ is Riemann integrable precisely when it is Darboux integrable, and in such cases, the values of the integrals calculated by both methods are identical. Demonstrate their equivalence with the function $f(x) = x^3$ on the interval [0, 2]. 7

2. (a) Prove that if *f* is a bounded function on $[a, b]$, then $L(f) \leq U(f)$. Consider the polynomial function $f(x) = x^2 - 4x + 3$ defined on the interval [0, 5]. Divide the interval into five equal subintervals to illustrate the theorem $L(f) \leq U(f)$. Calculate the exact integral of *f* over [0, 5] using basic calculus techniques. 3+4=7

   (b) Let *f* be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and $\int_a^b f = \int_a^b g$. 7

UNIT–II

3. (a) If f is integrable on $[a, b]$, then prove that $|f|$ is integrable on $[a, b]$ and $\left|\int_a^b f\right| \le \int_a^b |f|$. Justify whether the converse is true. 5+2=7

(b) Apply the theorem on change of variables to integrate: 3½×2=7

(i) $\int_0^1 x\sqrt{1-x^2}\,dx$ (ii) $\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

4. (a) Let f be a function defined on $[a, b]$. If $a<c<b$ and f is integrable on $[a, c]$ and on $[c, b]$, then prove that f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$ 7

(b) Let g be a one-to-one differentiable function on an open interval I. Then $J=g(I)$ is an open interval, and the inverse function g^{-1} is differentiable on J. Show that

$\int_a^b g(x)dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du = b.g(a) - a.g(a)$ for $a, b \in I$. Use this result to integrate $\int_0^{\frac{1}{2}} \sin^{-1} x dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$. 3½+3½=7

UNIT–III

5. (a) Show that $\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$. 6

(b) From definition, evaluate the following: 2×4=8

(i) $\int_0^\infty x^{-\frac{3}{2}} dx$ (ii) $\int_{-\infty}^\infty x e^{-x^2} dx$

(iii) $\int_0^1 x^{-\frac{2}{3}} dx$ (iv) $\int_{-\infty}^\infty \frac{dx}{x^2+4}$

6. (a) Test the following integrals for convergence: 4×2=8

(i) $\int_0^1 \frac{e^x dx}{\sqrt{1-\cos x}}$ (ii) $\int_0^\infty \frac{x \sin x\, dx}{a^2+x^2}$

(b) Show that: 3×2=6

(i) $\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}$

(ii) $\int_0^1 x^3 \left(1-x^2\right)^{3/2} dx = \frac{2}{63}$

UNIT–IV

7. (a) If the functions f_n be continuous on an interval I and that the sequence (f_n) converges uniformly to a function f on I, then prove that f is continuous on I. Give an example to illustrate. 4½+2½=7

(b) State Weierstrass M-test. Prove using the Weierstrass M-test that the series $\sum_{n=1}^\infty \frac{\cos(nx)}{2^n}$ converges uniformly on $\mathbb{R}$. 1+6=7

8. (a) Discuss the pointwise and uniform convergence of the sequence (f_n) where $f_n(x) = \frac{x^n}{1+x^n}. x \in [0, \infty)$. 7

(b) Prove that $\sum_{n=1}^\infty \frac{\sin nx}{n^2}$ is continuous on $\mathbb{R}$. 7

UNIT–V

9. (a) For each of the following power series, find the radius of convergence and determine the exact interval of convergence. 4×2=8

(i) $\sum_{n=0}^\infty \frac{2^n x^n}{n!}$ (ii) $\sum_{n=0}^\infty \sqrt{n} x^n$

(b) Discuss the characteristics of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ by means of Abel's theorem. 6

10. (a) Explain the Cauchy-Hadamard theorem and use it to find the radius of convergence for the power series $\sum_{n=0}^{\infty} n^2 x^n$. 1+6=7

(b) Consider the power series $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with radius of convergence $R > 0$. Prove that the series obtained by term-by-term integration of $g(x)$ also converges for $|x| < R$. Determine the radius of convergence of the integrated series and express its sum function in terms of $g(x)$. 7