2024

B.A./B.Sc.

Second Semester

CORE – 4

MATHEMATICS

Course Code: MAC 2.21

(Differential Equations)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Find a general solution of $(x^2+1)\dfrac{dy}{dx}+3xy=6x$. 5

   (b) Consider the differential equation $(4x+3y^2)dx+(2xy)dy=0$. Find an integrating factor of the form x^n and hence solve. 5

   (c) Solve the equation $\dfrac{dy}{dx}=\dfrac{3y^2-x^2}{2xy}$. 4

2. (a) Solve the equation $(5x+2y+1)dx+(2x+y+1)dy=0$ by making a suitable transformation. 6

   (b) Verify that $y(x)=\tan(x^3+C)$ satisfies the differential equation $y'=3x^2(y^2+1)$. Then determine a value of the constant C so that $y(x)$ satisfies the initial condition $y(0)=1$. 4

   (c) Solve the differential equation $\dfrac{dy}{dx}=(x+y+3)^2$. 4

UNIT–II

3. (a) Suppose a population can be modelled using the differential equation $\dfrac{dX}{dt}=0.2X-0.001X^2$, with an initial population size of $x_0=100$ and a time step of 1 month. Find the predicted population after 2 months. 5

(b) The mass of a radioactive isotope decays at a rate proportional to the mass of the isotope present at that instant. The half-life of the isotope is 12 days. Show that the proportion of the original amount of the isotope left after a period of 30 days is $\frac{1}{8}\sqrt{2}$. 5

(c) Suppose that a certain drug is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 mg of the drug per kg of the dog's body weight. Suppose also that the drug is eliminated exponentially from the dog's bloodstream, with a half-life of 5 hours. What single dose should be administered in order to anesthetize a 50 kg dog for 1 hour? 4

4. (a) A pond initially containing 500,000 gallons of unpolluted water has an outlet that releases 10,000 gallons of water per day. A stream flows into the pond at 12,000 gallons per day containing water with a concentration of 2 grams per gallon of a pollutant. 5+3=8

(i) Find a differential equation that models this process and solve to find a function that shows the concentration of pollutant in the pond at time *t*.

(ii) Determine the amount and the concentration of pollutant in the pond after 10 days.

(b) Suppose that a mineral body formed in an ancient cataclysm originally contained the uranium isotope ^{238}U (which has a half-life of 4.51×10^9 years) but no lead, the end product of the radioactive decay of ^{238}U. If today the ratio of ^{238}U atoms to lead atoms in the mineral body s 0.9, when did the cataclysm occur? 6

UNIT–III

5. (a) Verify that $y_1(x) = x^3$ is one solution of the equation $x^2 y'' + xy' - 9y = 0$. Then derive by reduction of order the second linearly independent solution of the given equation. 5

(b) Solve the initial-value problem

$$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0, y(0) = 3, y'(0) = -1.$$ 5

(c) Let $y_1(x)$ be a solution of $y''+p(x)y'+q(x)y=0$ (on an interval I where p and q are continuous functions of x). If $y_2(x)=v(x)y_1(x)$ is a second linearly independent solution of the given equation, use the fact that $y_1(x)$ is a solution to deduce that $y_1v''+(2y_1'+py_1)v'=0$. 4

6. (a) Show that $y_1=\sin x^2$ and $y^2=\cos x^2$ are linearly independent functions, but their Wronskian vanishes at $x=0$. Why does this imply that there is no differential equation of the form

$y''+p(x)y'+q(x)y=0$, with both p and q continuous everywhere, having both y_1 and y_2 as solutions? 6

(b) Show that $y_1=\sin 2x$ and $y_2=2\sin x\cos x$ are solutions of the differential equation $y''+4y=0$. Explain why $y=y_1+y_2$ is not the general solution of the given differential equation. 4

(c) Solve $y''-2iy'+3y=0$. 4

UNIT–IV

7. (a) Solve the initial-value problem

$y''+3y'+2y=e^x;\ y(0)=0,\ y'(0)=3$. 5

(b) Solve the differential equation

$$\frac{d^4y}{dx^4}+6\frac{d^3y}{dx^3}+15\frac{d^2y}{dx^2}+20\frac{dy}{dx}+12y=0.$$ 5

(c) Find the general solution of $x^2\dfrac{d^2y}{dx^2}-6x\dfrac{dy}{dx}+10y=3x^4+6x^3$,

given that $y=x^2$ and $y=x^5$ are linearly independent solutions of the corresponding homogeneous equation. 4

8. (a) Find a complementary function of the homogeneous differential equation $x^2y''-4xy'+6y=0$ and hence solve the differential equation $x^2y''-4xy'+6y=x^3$. 6

(b) Solve the given differential equation using method of variation of parameters: $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=\dfrac{e^{-x}}{x}$. 5

(c) For the given differential equation, set up an appropriate form of a particular integral y_p, but do not determine the values of the coefficients: $(D-1)^3(D^2-4)y = xe^x + e^{2x} + e^{-2x}$.

Here $D \equiv \dfrac{d}{dx}$. 3

UNIT–V

9. Let $X(t)$ be a harmful insect population that under natural conditions is held somewhat in check by a benign predator insect population $Y(t)$. Assume that $X(t)$ and $Y(t)$ satisfy the predator-prey equations

$$\frac{dX}{dt} = 200X - 4XY, \quad \frac{dY}{dt} = -150Y + 2XY$$

(a) Describe the nature of interaction-competition, cooperation or predation, between the two insect populations and determine what nonzero X and Y populations can coexist. 3

(b) Suppose an insecticide is sprayed on the two insect populations and that the per-capita death rate p due to the insecticide is the same for both prey and predator.

(i) Modify the model given above to develop a new mathematical model that incorporates the impact of the insecticide on the system and use chain rule to find a relationship between the two insect populations. 4

(ii) Find all equilibrium points of this new system. 2

(iii) Find the predator fraction of the total average population. What conclusion can you draw about the use of insecticide in this particular case? 5

10. (a) The following battle model represents two armies where both are exposed to aimed fire, and for one of the armies (red) there is significant loss due to desertion (at a constant rate c). The numbers of soldiers, R and B, satisfy the differential equations $\dfrac{dR}{dt} = -a_1B - c$, $\dfrac{dB}{dt} = -a_2R$, where a_1, a_2 and c are positive constants.

(i) If the initial number of red soldiers is r_0, and the initial number of blue soldiers is b_0, find a relationship between B and R. 3

(ii) For $a_1 = a_2 = c = 0.01$, give a sketch of typical phase-plane trajectories and deduce the direction of travel along the trajectories, providing reasons. 5

(b) A model for the spread of a disease, where all infectives recover from the disease and become susceptibles again, is given by the pair of differential equations $\frac{dS}{dt} = -\beta SI + \gamma I$, $\frac{dI}{dt} = \beta SI - \gamma I$, where β and γ are positive constants, $S(t)$ denotes the number of susceptibles and $I(t)$ denotes the number of infectives at time t.

(i) Use the chain rule to find a relationship between the number of susceptibles and the number of infectives given the initial number of susceptibles is s_0 and there was initially only one infective. 3

(ii) Draw a sketch of typical phase-plane trajectories. Deduce the direction of travel along the trajectories. 3
