2024

B.A./B.Sc.

Second Semester

CORE – 3

MATHEMATICS

Course Code: MAC 2.11

(Real Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Also, show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$. 5

   (b) Show that the set $\mathbb{N} \times \mathbb{N}$ is denumerable. 5

   (c) Let a, b, c be any elements of $\mathbb{R}$, prove that 4

   (i) If $a > b$ and $b > c$, then $a > c$.

   (ii) If $a > b$, then $a + c > b + c$.

2. (a) Find all $x \in \mathbb{R}$ that satisfy the inequality $4 < |x+2| + |x-1| < 5$. 4

   (b) Let $S = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$. Find $\inf S$ and $\sup S$. 3

   (c) Prove that the unit interval $[0,1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$ is not countable. 5

   (d) If $a \in \mathbb{R}$, show that $|a| = \sqrt{a^2}$. 2

UNIT–II

3. (a) Suppose that x and y are any two real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. 5

(b) State Bolzano-Weierstrass theorem for bounded sets. Prove that the set $S = \left\{3^n + \frac{1}{3^n} : n \in \mathbb{N}\right\}$ has no limit point. Does it contradict the theorem? 5

(c) Find the derived sets of the following sets: 2×2=4

(i) $\left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

(ii) $\left\{2^n + \frac{1}{2^n} : n \in \mathbb{N}\right\}$

4. (a) Prove that 0 is the only limit point of the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ 6

(b) Given S and T are subsets of real numbers $\mathbb{R}$. Prove or disprove: $(S \cap T)' = S' \cap T'$. 4

(c) Show that if $x \in \mathbb{R}$ is a limit point of $A \cap B$, then x is a limit point of A and B. Is the converse true? Justify your answer. 4

UNIT–III

5. (a) Prove that $\lim_{n\to\infty}(n^{1/n}) = 1$. Use this result to show that

$$\lim_{n\to\infty} \frac{1}{n}[1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}] = 1.$$ 5

(b) Let $x_n := \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ for each $n \in \mathbb{N}$. Prove that (x_n) is monotonically increasing and bounded, and that $\lim x_n \leq 2$. 5

(c) Using ε-definition, show that $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right) = 0$. 4

6. (a) If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to $z \neq 0$, then prove that the quotient sequence X / Z converges to x / z. 7

(b) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(c) Every bounded sequence is convergent. Justify. 2

UNIT–IV

7. (a) State and prove the Cauchy convergence criterion. 8

(b) Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences then $(x_n + y_n)$ is also Cauchy sequence. 4

(c) Give one example of an unbounded sequence that has a convergent subsequence. 2

8. (a) Show that the sequence $(x_n) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$ is not a Cauchy sequence. Is it convergent? 5

(b) Prove that if the subsequence (x_{2n-1}) and (x_{2n}) converges to the same limit l, then (x_n) also converges to l. 4

(c) Prove that $(n \sin n)$ is properly divergent. 3

(d) A sequence (x_n) converges to l if every subsequence of (x_n) converges to l. Is the converse true? Justify. 2

UNIT–V

9. (a) Show that the following series is not convergent. 4

(i) $\sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^{1/n}$

(ii) $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots$

(b) Using comparison test, examine the convergence of infinite series

$$\sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1}) \; .$$ 4

(c) Using integral test, show that the series $\sum_{n=2}^{\infty}\frac{1}{(n\log n)^p}$ is convergent if $p>1$ and divergent if $0<p\leq 1$. 6

10. (a) Test the convergence of the series $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\ldots,\ x>0.$ 5

(b) Prove that the series $\sum_{n=1}^{\infty}\frac{(-1)^n \log n}{n^2}$ is convergent. Is it absolutely convergent? 5

(c) Prove that if $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. 4