2023

M.Sc.

Fourth Semester

DISCIPLINE SPECIFIC ELECTIVE – 03

MATHEMATICS

Course Code: MMAD 4.11

(Differential Geometry of Manifolds)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) (i) If X is a tangent vector, then show that $X(c)=0$, where c is a constant. 2

(ii) If X^i and $\overline{X}^i$ are the components of a tangent vector $X \in T_p(M)$ relative to the basis $\{0_i\}$ and $\{\overline{0}_i\}$, then show that

$$\overline{X}^i = \frac{\partial \overline{x}}{\partial x^j} X^j \text{ and } X^i = \frac{\partial x^{-i}}{\partial \overline{x}^j} \overline{X}^j .$$ 1

(iii) Prove that the general linear group $Gl(n,R)$ is a differentiable manifold of dimension n^2. 3

(b) If $\phi : M_n \to M_m$ and $\psi : M_m \to M_p$ are differentiable maps and if x lies in the domain of $\psi \circ \phi$, then show that

(i) $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ 1

(ii) $\phi^*(A+B) = \phi^* A + \phi^* B$ and $\phi^*(A \otimes B) = \phi^* A \otimes \phi^* B$. 3

(c) Discuss in detail one–parameter group of transformations. 4

2. (a) Let $\phi = 6dx \wedge dy + 27dx \wedge dz$ and $\psi = dx + dy + dz$, then compute $\phi \wedge \psi$. Again, compute

$(5dx + 3dy + dz) \wedge (6dx + 2dy + 3dz) \wedge (3dx + 2dy)$ 3+3=6

(b) Show that the exterior product is associative, distributive, homogeneous and anticommutative. 8

UNIT–II

3. (a) (i) Prove that if a Lie group G has dimension n, then its Lie algebra Lie(G) is also of dimension n.

(ii) Prove that the set $C^* = GL(l, C)$ is the multiplicative group of nonzero complex numbers.

4+3=7

(b) Give in detail the definition of general linear group. Prove that if $\phi : G \to G$ is the diffeomorphism of the Lie group G defined by $\phi(a) = a^{-1}, a \in G$, then w is a left invariant form if and only if $\phi^* w$ is right invariant. 5+2=7

4. (a) What do you mean by Lie transformation groups? 5

(b) Define linear frame bundle and bundle homomorphism. 5+4=9

UNIT–III

5. (a) Write the definition of parallelism and use it to prove the following:

(i) T_σ is a diffeomorphism

(ii) $R_g T_\sigma = T_\sigma R_g$

(iii) T_σ is independent of the parameterization

(iv) If γ and σ are two such curves with $\sigma(0) = \gamma(1)$, then $T_{\gamma\sigma} = T_\sigma \circ T_\gamma$, where T_σ is the parallel transformation of the fibre along σ. 10

(b) If ϕ is a pseudotensorial r–form on L of type (P,V), then show that the form ϕh defined by $\phi h(X_1, X_2, ..., X_r) = \phi(hX_1, hX_2, ..., hX_r)$, X_i's$\in T_z$ is a tonsorial form of type (P,V). 4

6. Prove that:

(a) The law of transformation of the coefficients of connection from the basis $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial \overline{x}^i}$ is $\overline{\Gamma}^p_{qr} = \overline{\Gamma}^i_{jk} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^q} \frac{\partial x^k}{\partial \overline{x}^r} + \frac{\partial^2 x^h}{\partial \overline{x}^q \partial \overline{x}^r} \frac{\partial \overline{x}^p}{\partial x^h}$ 4

(b) The symmetric part of coefficients of a connection are coefficients of connection but the skew symmetric part are components of tensors. 4

(c) The Lie derivative is derivative, i.e. 3+3=6

(i) $L_X(P+Q) = L_X P + L_X Q$

(ii) $L_X(P \otimes Q) = (L_X P) \otimes Q + P \otimes L_X Q$

UNIT–IV

7. (a) Show that the curvature tensors satisfy: 1+3+6=10

(i) $K(X, Y, Z) = -K(Y, X, Z)$

(ii) $K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0$

(iii) $(D_X K)(Y, Z, U) + (D_Y K)(Z, X, U) + (D_Z K)(X, Y, U) = 0$

(b) Use Bianchi's identity to show that $2(div\, R)(X) = Xr$. 4

8. (a) Define sectional curvature and prove that a manifold of constant curvature is necessarily an Einstein manifold. 10

(b) State and prove Beltrami's theorem. 4

UNIT–V

9. (a) Discuss in detail the normal vector field of a manifold and prove that the normal vector field $V(X, Y)$ is bilinear in X and Y. 6+2=8

(b) If $\overline{\Gamma}$ is a Riemannian connection of a Riemannian manifold related to the metric $\overline{g}$, than prove that the connection Γ induced on a submanifold is also a Riemannian connection of the submanifold relative to the induced metric g. 6

10. (a) State and prove Weingarten equations. 2+8=10

(b) Show that a necessary and sufficient condition that a submanifold is totally umbilical is that $hV=kVg$. Also, show that the principal curvatures are equal at each umbilical point of a hypersurface. 4