2023

M.Sc.

Fourth Semester

CORE – 12

MATHEMATICS

Course Code: MMAC 4.21

(Rings & Modules)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Prove that every finite integral domain is a field. 4

   (b) If R is any ring, prove that the set $M_n(R)$ of $n \times n$ matrices with entries in R is a ring under matrix addition and matrix multiplication. 5

   (c) If R is a ring, define R^{op} and prove that it is a ring. 5

2. (a) If G is an additive abelian group, prove that the set of all group endomorphisms is a ring under pointwise addition and composition of maps. 5

   (b) Define local ring and also give some examples of local rings. Prove that a ring R with 1 is a local ring if and only if it has a unique maximal ideal, maximal simultaneously as a left/right/two-sided ideal. 4

   (c) If I and J are ideals of a ring R, prove that $I+J$ and IJ are also ideals of R. 5

UNIT–II

3. (a) If R is a ring, prove that M is a left R-module if and only if M is a right R^{op}-module, where R^{op} is the ring opposite to R. 5

   (b) If R is a ring and N is a submodule of an R-module M, prove that the quotient group M/N is an R-module. 5

(c) If M and N are R-modules and $f : M \to N$ is an R-linear map, prove that kernel of f is a submodule of M and image of f is a submodule of N. 4

4. (a) If M is an abelian group, R is a ring and $End_Z(M)$ is the ring of all additive endomorphisms of M, prove that M is a right R-module if and only if there exists an anti-homomorphism $\Psi' : R \to End_Z(M)$. 5

(b) If N and K are submodules of an R-module M, prove that

$$\frac{N+K}{N} \approx \frac{K}{K \cap N}.$$ 5

(c) If P and Q are submodules of an R–module M, prove that $P+Q$ is a submodule of M, containing both P and Q. 4

UNIT–III

5. (a) Define the following

(i) Direct sum of two modules

(ii) Direct summand of a module

If M and N are submodules of a module P over a ring R, prove that $M \cap N = (0)$ if and only if every element $t \in M + N$ can be uniquely written as $t = p + q$, where $p \in M, q \in N$. 4

(b) Define free module and prove that every vector space is a free module. 5

(c) Prove that every free module is a projective module and justify that the converse is not true by an example. 5

6. (a) Define a torsion module and a torsion–free module. If R is an integral domain and M is an R–module, show that the set T of the torsion elements of M is a submodule of M and the quotient module M/T is torsion-free. 5

(b) Prove that the functor $Hom_R(P,)$ is left exact and deduce the condition for P to be a projective module. 6

(c) Prove that any abelian group which has a non-trivial element of finite order cannot be free. 3

UNIT–IV

7. (a) Define simple module and prove that if A and B are simple modules over a ring R, then any R-linear map $f : A \to B$ is either the zero map or an isomorphism. 4

(b) If $\{M_i\}_{i \in I}$ is a family of R-modules, prove that $\prod_{i \in I} M_i$ is injective if and only if each M_i is injective. 6

(c) Define divisible module and also give an example. If M is a divisible R-module and N is a submodule of M, prove that M/N is a divisible module. 4

8. (a) Prove that an R-module Q is injective if and only if every exact sequence of the form $0 \to Q \to M \to M'' \to 0$ splits. 6

(b) Prove that every $\mathbb{Z}$-module can be embedded in an injective $\mathbb{Z}$-module. 4

(c) Define semi-simple module and prove that every homomorphic image of a semi-simple module is semi-simple. 4

UNIT–V

9. (a) Define Artinian and Noetherian modules and also give one example of each. Prove that if a module M is such that it has a submodule N with both N and M/N Noetherian, then M is Noetherian. 7

(b) Define composition series of a non-zero module and module of finite length. Prove that a module is of finite length if and only if it is both Artinian and Noetherian. 7

10. (a) Define nil radical and Jacobson radical of a ring and also give one example of each. State and prove Nakayama Lemma. 8

(b) Prove that every Artinian ring is Noetherian. 6
