2023
M.Sc.
Second Semester
CORE – 08
MATHEMATICS
Course Code: MMAC 2.41
(Complex Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Suppose that $z_n = x_n + iy_n (n = 1, 2, 3, ...)$ and $z = x + iy$. Prove that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. 5

(b) Prove that a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ represents a continous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. 5

(c) Show that the following series for $\sin z$ is absolutely and uniformly convergent for all values of z: 4

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... + \frac{(-1)^n z^{2n+1}}{(2n+1)!+...}$$

2. (a) Let C denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, and let $g(z)$ be any function that is continous on C. Prove that the series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C. 5

(b) If a series $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to $f(z)$ at all points in some annular domian about z_0, then prove that it is the Laurent series expansion for f in powers of $z - z_0$ for that domian. 5

(c) Show that if $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} \overline{z}_n = \overline{S}$ 4

UNIT–II

3. (a) Give two Laurent series expansions in powers of z for the function $f(z) = \frac{1}{z^2(1-z)}$, and specify the regions in which those expansions are valid. 4

(b) Prove that an isolated point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and nonzero at z_0. 5

(c) Determine the residues of the following functions: 2½×2=5

(i) $\frac{z^2}{z^2 + a^2}$ at $z = ia$

(ii) $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z = 1$

4. (a) Evaluate the integral $\int_c \frac{5z-2}{z(z-1)}$ when, C is the circle $|z| = 2$, described counterclock-wise. 4

(b) Suppose that two functions p and q are analytic at a point z_0, satisfying $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$. Prove that the quotient $\frac{p(z)}{q(z)}$ has a simple pole at z_0 and $\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$ 5

(c) Represent a function $f(z) = \frac{z}{(z-1)(z-3)}$ by a series of positive and negative powers of $(z-1)$ which converges to $f(z)$ in the annular region $0 < |z-1| < 2$. 5

UNIT–III

5. (a) Use residue to find the Cauchy principle value of the integral

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)}$$ 7

(b) Derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax)-\cos(bx)}{x^2}dx = \frac{\pi}{2}(b-a), \quad (a \geq 0, b \geq 0)$$ 7

6. (a) Use residue to evaluate the definite integral $\int_0^{2x} \frac{d\theta}{1+a\cos\theta}$ 7

(b) Find the Cauchy principle value of the integral $\int_{-\infty}^{\infty} \frac{x\sin xdx}{x^2+2x+2}$ 7

UNIT–IV

7. (a) Determine the number of zeroes, counting multiplicities of the polynomials

(i) z^4+3z^3+6

(ii) $z^4-2z^3+9z^2+z-1$

inside the circle $|z|=2$. 2½+2½=5

(b) Suppose that a function f is analytic inside and on a positively oriented simple closed contour *C* and that ithas no zero on *C*. Show that if f has *n* zeroes $z_k (k=1,2,...,n)$ inside *C*, where z_k is of multiplicity m_k, then $\int_C \frac{zf'(z)}{f(z)}dz = 2\pi i \sum_{k=1}^{n} m_k z_k$ 5

(c) Define winding number. What is the argument principle? 4

8. (a) Using Rouche's theoram, prove that any polynomial $P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ $(a_n \neq 0)$, where $n \geq 1$, has precisely *n* zeroes, counting multiplicities. 6

(b) Let *C* denote the unit circle $|z|=1$, described in the positive sense. Determine the value of $\Delta_C \arg f(z)$ when 2×2=4

(i) $f(z)=\frac{(z^3+2)}{z}$

(ii) $f(z) = \dfrac{(2z-1)^7}{z^3}$

(c) Show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. 4

UNIT–V

9. (a) Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n (n = 1, 2, 3, ...)$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point. 5

(b) Find a linear fractional transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $-i, 1, i$ in the w-plane respectively. 5

(c) Find the image of the half plane $y > 1$ under the transformation $w = (1-i)z$, where $z = x + iy$. 4

10. (a) Find the image of the half plane $x \geq c_1 (c_1 > 0)$ under the transformation $w = \dfrac{1}{z}$ where $z = x + iy$. 5

(b) Show that there is only one linear fractional transformation that maps three given distinct points z_1, z_2 and z_3 in the extended z-plane onto three specified distinct points w_1, w_2 and w_3 in the extended w-plane. 5

(c) Show that the transformation $w = \sin z$ is conformal at all points except $z = \dfrac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$. 4