2023
M.Sc.
Second Semester
CORE – 07
MATHEMATICS
Course Code: MMAC 2.31
(Number Theory)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT – I

1. (a) Show that $n\,|\,(n-1)!$ for all composite integers greater than 4. 5

   (b) If m is odd, show that 2, 4, 6, ... , $2m$ is a complete residue system modulo m. 4

   (c) Find all the solutions of $57x \equiv 87 \pmod{105}$. 5

2. (a) Prove/disprove: If $a^3\,|\,b^3$, then $a\,|\,b$ 4

   (b) For any integer n, show that $42\,|\,n^7 - n$ 5

   (c) Solve: $x^3 + 2x - 3 \equiv 0 \pmod 9$ 5

UNIT – II

3. (a) State and prove Euler's criterion. 7

   (b) Derive that formula for the Legendre symbol $\left(\frac{2}{p}\right)$, where p is an odd prime. 7

4. (a) If p and q are distinct odd primes, prove that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\{(p-1)/2\}\{(q-1)/2\}}$$ 7

(b) If Q is odd, show that the Jacobi symbol $\left(\frac{-1}{Q}\right) = (-1)^{(Q-1)/2}$ 7

UNIT – III

5. (a) Show that the Diophantine equation $x^4 - y^4 = z^2$ has no solution for positive integers x, y, z. 9
   (b) Prove that in a primitive Pythagorean triple x, y, z, the product xy is divisible by 12 and the product xyz is divisible by 60. 5

6. (a) Prove that $y^2 = x^3 + 7$ has no solution in integers. 7
   (b) Prove that in a primitive Pythagorean triple x, y, z, not more that one of x, y, or z can be a perfect square. 5
   (c) Give a Pythagorean triple whose terms form a geometric progression. 2

UNIT – IV

7. (a) Show that the greatest common divisor of two Fibonacci numbers is a Fibonacci number. 7
   (b) Show that the sum of the first n Fibonacci numbers with odd indices is given by $u_1 + u_3 + u_5 + \cdots + u_{2n-1} = u_{2n}$ 7

8. (a) Show that two consecutive Fibonacci numbers are relatively prime. 7
   (b) Show that the sum of the squares of the first n Fibonacci numbers is given by $u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2 = u_n u_{n+1}$ 7

UNIT – V

9. (a) Compute the convergents of the simple continued fraction $[1; 2, 3, 3, 2, 1]$ 5

(b) If $C_k = p_k / q_k$ denotes the kth convergent of the finite simple continued fraction $[1; 2, 3, 4, \ldots, n, n+1]$, show that

$$p_n = np_{n-1} + np_{n-2} + (n-1)\, p_{n-3} + \ldots + 3p_1 + 2p_0 + (p_0 + 1)$$ 9

10. (a) Evaluate p_k, q_k, and $C_k\ (k = 0, 1, \ldots 8)$ for the simple continued fraction $[1; 1, 2, 1, 2, 1, 2, 1, 2]$ 9

(b) By means of continued fractions, determine the general solutions of the Diophantine equation $19x + 51y = 1$ 5
