2023
M.Sc.
Second Semester
CORE – 06
MATHEMATICS
Course Code: MMAC 2.21
(Measure Theory)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Define a ring and a σ-ring . If $X = \mathbb{R}$, the real line, then show that the class $\mathcal{P} = \{[a,b) : a, b \in \mathbb{R}, a \leq b\}$ is a ring. 2+4=6

   (b) Prove that if $\mathcal{A}$ is an algebra of subsets of a set X then $\mathcal{M}(\mathcal{A}) = \mathcal{S}(\mathcal{A})$,that is, the smallest monotone class and the smallest σ-algebra containing algebra $\mathcal{A}$ are the same. 8

2. (a) Let X be a non-empty set and $\mathcal{C}$ be any class of subsets of a set X. Show that the intersection of all σ-algebras containing $\mathcal{C}$ is a σ-algebra and it is the smallest σ-algebra on X containing $\mathcal{C}$. 7

   (b) Show that a σ-ring is a monotone class and that a monotone ring is a σ-ring . (3½+3½=7)

UNIT–II

3. (a) Prove that if $\mathcal{A}$ is an algebra of subsets of X and $\mu : \mathcal{A} \to [0, \infty]$ a set function with $\mu(\emptyset) = 0$, then μ is countably additive if and only if μ is both finitely additive and countably subadditive. 7

   (b) Prove that if S^* the class of all μ^*-measurable subsets of X is an algebra of subsets of X , and μ^* restricted to S^* is finitely additive. 7

4. (a) Define outer measure μ^* on an algebra of subsets of a set X. Prove that the set function $\mu^* : P(X) \to [0, \infty]$ is countably additive.

2+6=8

(b) Let (E_n) be a sequence of measurable sets. Prove that if $E_1 \subseteq E_2 \subseteq ....$, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$. 6

UNIT–III

5. (a) Prove that the Lebesgue outer measure of an interval is equal to its length. 7

(b) Let $\mathcal{A}$ be an algebra of subsets of a set X and μ a measure on $\mathcal{A}$. Then prove that there exists a σ-algebra $\mathcal{S}^*$ (the σ-algebra of μ^*-measurable subsets of X) and a measure μ^* on $\mathcal{S}^*$ such that $\mathcal{A} \subseteq \mathcal{S}^*$ and $\mu^*(A) = \mu(A) \forall A \in \mathcal{A}$. 7

6. (a) Let $E \subseteq \mathbb{R}$. Prove that if E is a Lebesgue measurable set, then for every $\varepsilon > 0$, there exists an open set G_ε such that $E \subseteq G_\varepsilon$ and $\lambda^*(G_\varepsilon - E) < \varepsilon$. 7

(b) Let $A \in \mathcal{L}$ and $x \in \mathbb{R}$. Show that: 3½×2=7

(i) $A + x \in \mathcal{L}$, where $A + x := \{y + x : y \in A\}$

(ii) $-A \in \mathcal{L}$, where $-A := \{-y : y \in A\}$

UNIT–IV

7. (a) Prove that if f and g are measurable functions, then the three sets $\{x \in X : f(x) > g(x)\}$, $\{x \in X : f(x) \geq g(x)\}$ and $\{x \in X : f(x) = g(x)\}$ are all measurable. Moreover, show that the functions $f \vee g$, $f \wedge g$, f^+, f^- and $|f|$ are all measurable functions. 8

(b) Prove that if $\mu(X) < +\infty$ and $f_n \to f$ almost uniformly on X, then $f_n \to f$ almost everywhere. Give an example of a sequence that converges pointwise but not almost uniformly with respect to the Lebesgue measure λ. 4+2=6

8. (a) Let $(X, \mathcal{S}, \mu)$ be a complete measure space. Then prove the following statements:

(i) Given $f, g : X \to \mathbb{R}^*$, and if f is measurable then g is measurable.

(ii) If (f_n) is a sequence of measurable functions from X to $\mathbb{R}^*$ and if $f : X \to \mathbb{R}^*$ such that $f(x) = \lim_n f_n(x)$ holds for almost everywhere then f is measurable. 4+4=8

(b) Let $\mu(X) < +\infty$. Prove that if (f_n) converge almost everywhere to f, then (f_n) converges in measure to f. 6

UNIT–V

9. (a) For $s, s_1, s_2 \in \mathsf{L}_0^+$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, prove that the following statements hold:

(i) $0 \leq \int s d\mu \leq +\infty$

(ii) $\alpha s \in \mathsf{L}_0^+$ and $\int (\alpha s) d\mu = \alpha \int s d\mu$

(iii) $s_1 + s_2 \in \mathsf{L}_0^+$ and $\int (s_1 + s_2) d\mu = \int s_1 d\mu + \int s_2 d\mu$

(iv) For $E \in \mathcal{S}$, we have $s\chi_E \in \mathsf{L}_0^+$, and set function v on E is given by $\int s\chi_E d\mu$ is a measure on $\mathcal{S}$. Also, $v(E) = 0$ whenever $\mu(E) = 0$, $E \in \mathcal{S}$. 1+1+2+4=8

(b) For $f \in \mathsf{L}$, prove that $f \in L_1(\mu)$ if and only if $|f| \in L_1(\mu)$. Further, in either case show that $\left|\int f d\mu\right| \leq \int |f| du$. 3+3=6

10. (a) State and prove the monotone convergence theorem. 1+7=8

(b) Let $f \in L_1(\mu)$ and $E \in \mathcal{S}$. Show that $\chi_E f \in L_1(\mu)$, where $\int_E f du := \int \chi_E f du$. Further, if $E, F \in \mathcal{S}$ are disjoint sets, show that $\int_{E \cup F} = \int_E f d\mu + \int_F f d\mu$. 3+3=6