2023

B.A./B.Sc.

Second Semester

GENERIC ELECTIVE – 2

MATHEMATICS

Course Code: MAG 2.11

(Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If A and B are two bounded subsets of real numbers $\mathbb{R}$, then prove that $A \cap B$ is also bounded. 3

(b) If $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ then, show that

$$\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e.$$ 3

(c) Using ε-definition, show that the sequence $f_n = \sqrt{n+1}-\sqrt{n}$, $\forall n \in \mathbb{N}$ is convergent. 5

(d) Show that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is monotonically increasing and bounded above. 3

2. (a) Show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$$ 3

(b) If $a, b \in \mathbb{R}$, then prove that $|a+b| \le |a|+|b|$. 3

(c) Show that the sequence $\{S_n\}$, defined by the recursion formula $S_n = \sqrt{3S_n}, S_1 = 1$ converges to 3. 6

(d) Find the n^{th} term of $\left\{\sqrt{\frac{1}{4}}, \sqrt{\frac{2}{6}}, \sqrt{\frac{3}{8}}, \ldots\right\}$. 2

UNIT–II

3. (a) Use integral test to show that the series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. 6

(b) Using comparison test, show that the series $\sum \frac{bn - a}{bn^2 + a^2}$, where *a*, *b* are constant is divergent. 3

(c) Test for absolute convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{n^2}{(n+1)!}$$ 5

4. (a) Prove by comparison test that the series

$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \cdots$ diverges for $p > 0$. 6

(b) Show that the series $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots$ converges for $0 \leq x < 1$ and diverges for $x \geq 1$. 5

(c) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n + 1}$ 3

UNIT–III

5. (a) Find the equation whose roots are 3, 1, –4 and –5. 2

(b) Show that the roots of the equation $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}$, where $a>b>c>0$ are real. 6

(c) Prove that every equation of n^{th} degree has n roots 6

6. (a) Solve $x^3+6x^2-12x+32=0$ using Cardan's method. 6

(b) Apply Descartes' rule of sign to discuss the nature of roots of the equation $x^4+16x^2+7x-11=0$. 4

(c) Find the condition that the cubic equation $x^3-px^2+qx-r=0$ should have its root in geometric progression. 4

UNIT–IV

7. (a) For the set of real numbers $\mathbb{R}$, we define binary operation $*$ as follows $a*b=2ab$.

(i) Is $*$ commutative? If not, give a counterexample.

(ii) Prove or disprove that $*$ is associative. 4

(b) Show that the cube root of unity $\{1, w, w^2\}$ forms a group under multiplication. 4

(c) Prove that the union of two subgroups of a group G is a subgroup of G if and only if one is contained in the other. 6

8. (a) For each $a\in(G,*)$, the inverse element a^{-1} is unique. 4

(b) Let $\mathbb{Z}$ be the set of all integers. An operation $*$ defined by $a*b=a+b-2,\ \forall a,b\in\mathbb{Z}$. Show that $(\mathbb{Z},*)$ is an abelian group. 5

(c) Let S be a set defined by $S=\left\{\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}, a\in\mathbb{R}\right\}$. Is S closed under matrix multiplication? 2

(d) Prove that $\{1, i, -1, -i\}$ is a cyclic group under the operation of product of two complex numbers. Also, find the generators. 3

UNIT–V

9. (a) Show that $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$ is an odd permutation. 2

(b) Show that the group $(1, 2, 3, 4, 5, 6, \times_7)$ is cyclic. How many generators are there? 4

(c) Prove that an infinite cyclic group has exactly two generators. 4

(d) Let $H = \{0, 3, 6\}$. Find all the left cosets of H in $(G, +_9)$. 4

10. (a) State and prove Lagrange's theorem. 4

(b) Find all the generators of $\mathbb{Z}_8$ and $\mathbb{Z}_{20}$. 4

(c) Find $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}^{-1}$ 2

(d) If H is any subgroup of G and $h \in H$, then prove that $Hh = H = hH$. 4
