2023

B.A./B.Sc.

Sixth Semester

DISCIPLINE SPECIFIC ELECTIVE – 4

MATHEMATICS

Course Code: MAD 6.21

(Differential Geometry)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Define torsion and find an expression of torsion. Also, prove the necessary and sufficient condition that a given plane to be a curve is that $\tau = 0$ at all points. 4+2=6

(b) State and prove Serret-Frenet formulae. 4

(c) Find the radius of curvature and torsion of the curve

$x = a(3t - t^3), y = 3at^2, z = a(3t + t^3)$. 4

2. (a) If there is a one-one correspondence between the points of two curves and the tangents at corresponding points are parallel, show that the principal normals are parallel and therefore also the binomial. Also, prove that $\dfrac{\chi_1}{\chi} = \dfrac{ds}{ds_1} = \dfrac{\tau_1}{\tau}$. 4

(b) Let C be the original curve and C_1 be the locus of the centre of curvature, then prove that

(i) the tangent to C_1 lies in the normal plane of the original curve C.

(ii) in case original curve C has constant curvature, then the curvature of C_1 is also constant and torsion of C_1 varies inversely as that of C. 7

(c) Prove that $(x''')^2 + (y''')^2 + (z''')^2 = \dfrac{1}{\rho^2\sigma^2} + \dfrac{1+(\rho')^2}{\rho^4} = \dfrac{1}{\rho^4} + \dfrac{R^2}{\rho^4\sigma^2}$

where dashes denote differentiation with respect to s. 3

UNIT–II

3. (a) If the normal at a point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the coordinate plane in G_1, G_2, G_3, then show that the ratios $PG_1 : PG_2 : PG_3$ are constant. 4

(b) State and prove the geometrical interpretation of the second fundamental form of metric. 5

(c) Show that the curves bisecting the angles between the parametric curves are given by $Edu^2 - Gdv^2 = 0$. 5

4. (a) State and prove Meusnier's theorem. 4

(b) Show that if there is a surface of minimum area passing through a closed space curve, it is necessarily a minimal surface. 6

(c) Find the principal directions and principal curvatures on the surface $x = a(u+v), y = b(u-v), z = uv$. 4

UNIT–III

5. (a) State and prove the normal property of geodesics. 6

(b) Prove that a geodesic is either a plane curve or a line of curvature, or it is both. 3

(c) Define geodesic tangent and find torsion of a geodesic in terms of principle curvature. 5

6. (a) Obtain the formula for geodesic curvature. 5

(b) Find geodesic curvature of the parametric curve $u = c$ with the metric $ds^2 = \lambda^2 du^2 + u^2 dv^2$. 4

(c) Define conformal mapping and find the necessary and sufficient conditions for the differentiable homomorphism to be conformal. 5

UNIT–IV

7. (a) Show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to rectangular cartesian transformation of coordinates. 4

(b) Define inner product and show that the inner product of the tensor A_r^p and B_t^{rs} is a tensor of rank three. 5

(c) If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, show that $C_{ij}+C_{ji}$ is a covariant tensor of the second order. In particular if $C_{ij}A^iA^j=0$, then show that $C_{ij}+C_{ji}=0$. 5

8. (a) Define symmetric and anti symmetric tensors and find the number of independent components of these tensors. 5

(b) Prove that if the components of a tensor vanishes in one coordinate system, they vanish identically in all coordinate systems. 3

(c) Prove that the set n^3 functions A^{ijk} from the components of a tensor if $A^{ijk}B_{ij}^{p}=C^{pk}$ provided that B_{ij}^{p} is an arbitrary tensor and C^{pk} a tensor. What happens if B_{ij}^{p} is symmetrical in *i* and *j*? 6

UNIT–V

9. (a) Show that: 1+3+2=6

(i) $[ij,k]+[jk,i]=\dfrac{\partial g_{ki}}{\partial x^j}=\partial_j g_{ki}$

(ii) $\dfrac{\partial g^{ik}}{\partial x^j}=-g^{hk}\begin{Bmatrix} i \\ hj \end{Bmatrix}-g^{hi}\begin{Bmatrix} k \\ hj \end{Bmatrix}$

(iii) $\begin{Bmatrix} i \\ ij \end{Bmatrix}=\dfrac{1}{2}\dfrac{\partial}{\partial x^j}\log g$

(b) Prove that the laws of transformations of Christoffel's symbols possess the group property. 5

(c) Suppose two unit vectors A^i and B^i are defined along a curve C such that their intrinsic derivatives along C are zero. Show that the angle between them is constant. 3

10. (a) Obtain the covariant derivative of a contravariant vector and write the rank of this mixed tensor. 5

(b) Show that the covariant differentiation of sums and products of tensors obey the same rules as ordinary differentiations. 5

(c) Prove that $divA^{ij} = \nabla_j A^{ij} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(A^{ij}\sqrt{g}\right) + A^{jk} \begin{Bmatrix} i \\ jk \end{Bmatrix}$.

What form does the above equation assume if A^{jk} is skew symmetric? 4